Brief solutions to Quiz 9

Dec 26, 2023:

1. (30 pts) Use any method to find the surface area of the torus (donut) generated by revolving the circle $(x-3)^2 + y^2 = 1$ around the y axis.

   Ans:

   See page 4-5 of Lecture 23 and page 2 of Homework 13 Solutions.

2. (30 pts) Solve for $y(x)$ from $\dfrac{dy}{dx} = x\sqrt{1-y^2}$ with $y(0) = \dfrac{1}{2}$.

   Ans:

$$\frac{dy}{\sqrt{1-y^2}} = x\,dx \implies \int_{s=\frac{1}{2}}^{y} \frac{ds}{\sqrt{1-s^2}} = \int_{t=0}^{x} t\,dt$$

$$\implies \sin^{-1}(s)\Big|_{s=\frac{1}{2}}^{y} = \frac{t^2}{2}\Big|_{t=0}^{x} \implies \sin^{-1}(y) = \frac{x^2}{2} + \frac{\pi}{6} \implies y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

   Remark:

   The solution $\sin^{-1}(y) = \dfrac{x^2}{2} + \dfrac{\pi}{6}$ or $y = \sin\left(\dfrac{x^2}{2} + \dfrac{\pi}{6}\right)$ is only valid on $|x| \le \sqrt{2 - \dfrac{\pi}{3}}$ since $|\sin^{-1}(y)| \le 1$ by definition. The full solution is

$$y(x) = \begin{cases} -1, & x \le -\sqrt{2-\dfrac{\pi}{3}} \\ \sin\left(\dfrac{x^2}{2} + \dfrac{\pi}{6}\right), & |x| \le \sqrt{2-\dfrac{\pi}{3}} \\ 1, & x \ge \sqrt{2-\dfrac{\pi}{3}} \end{cases}$$

3. (40 pts) Solve for $y(x)$ from $xy' - y = x \ln x$ on $x > 0$ with $y(1) = 2$.

Ans:

$$y' - \frac{y}{x} = \ln x$$

. Multiply by the integration factor $e^{\int \frac{-1}{x}} = \frac{1}{x}$,

$$\Longrightarrow \frac{y'}{x} - \frac{y}{x^2} = \frac{\ln x}{x} \Longrightarrow \left(\frac{y}{x}\right)' = \frac{\ln x}{x}$$

$$\Longrightarrow \int_1^x \left(\frac{y(t)}{t}\right)' dt = \int_1^x \frac{\ln t}{t} dt = \int_1^x \ln t \, d(\ln t) = \int_{t=1}^x d\left(\frac{\ln^2 t}{2}\right)$$

$$\Longrightarrow \frac{y(x)}{x} - \frac{y(1)}{1} = \frac{\ln^2 x}{2} - 0 \Longrightarrow y(x) = 2x + \frac{x \ln^2 x}{2}$$

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