Brief solutions to Quiz 7

Nov 28, 2023:

1. (30 pts) Find the point on $y = x^{\frac{3}{2}}$, $\underline{x \geq 0}$ that is closest to $(4, 0)$. Explain why the answer you have is actually a global minimum on $\{x \geq 0\}$.

   Ans:

   Let $d =$ "distance from (x, y) to $(4, 0)$". Minimizing d is the same as minimizing d^2.

   Let $f(x) = d^2$ on $\{y = x^{\frac{3}{2}}\}$, then

   $$f(x) = d^2 = (x-4)^2 + y^2 = (x-4)^2 + x^3 = x^3 + x^2 - 8x + 16.$$

   $$f'(x) = 3x^2 + 2x - 8 = (x+2)(3x-4)$$

   Therefore, $f'(x) \geq 0$ on $\{x \geq \frac{4}{3}\}$ and $f'(x) \leq 0$ on $\{0 \leq x \leq \frac{4}{3}\}$.

   That is, f is decreasing on $\{0 \leq x \leq \frac{4}{3}\}$ and increasing on on $\{x \geq \frac{4}{3}\}$.

   Thus the global minimum occurs at $x = \frac{4}{3}$ and the minimum is $\sqrt{f(\frac{4}{3})} = \frac{16}{3\sqrt{3}}$.

2. (30 pts) Solve for $y(x)$ on $x > 0$ from $y''(x) = \frac{1}{x^2}$, $y'(1) = 2$, $y(1) = 0$.

   Ans:

   $y'(x) = y'(1) + \int_1^x y''(t)dt = y'(1) + \int_1^x t^{-2}dt = 2 + (-t^{-1})\Big|_1^x = -\frac{1}{x} + 3$

   $y(x) = y(1) + \int_1^x y'(t)dt = y(1) + \int_1^x \big(-\frac{1}{t} + 3\big)dt = 0 + \big(-\ln|t| + 3t\big)\Big|_1^x = -\ln x + 3x - 3$

3. (30 pts) Express $\int_0^2 x^2\, dx$ as a limit of Riemann sum (with uniformly spaced partition, and c_k of your choice) and find the limit. (Hint: first write down the uniform partition on $[0, 2]$)

   Ans:

   $P = \{0 = x_0 < x_1 < \cdots < x_n = 2\}$. $x_k = \frac{2k}{n}$. $\Delta x_k = \frac{2}{n}$.

   $$\int_0^2 x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{2k}{n})^2 \frac{2}{n} = \lim_{n\to\infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{8}{3}$$