Calculus I, Fall 2023

Brief solutions to Quiz 5

Nov 14, 2023:

1. (30 pts) Let
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
Is f differentiable at $x = 0$? Is f twice differentiable at $x = 0$? Start with definitions of $f'(0)$ and $f''(0)$ and explain.

   Ans:

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x^2}) - 0}{x - 0} = 0 \quad \text{(Sandwich Theorem)}$$
$$f'(x) = 2x \sin(\frac{1}{x^2}) - \frac{2}{x} \cos(\frac{1}{x^2}), \quad x \neq 0$$
$$\lim_{x \to 0} f'(x) \quad \text{does not exist}$$

   So, f' is not continuous at $x = 0$, therefore not differentiable at $x = 0$ and
$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \quad \text{does not exist}$$

2. (20 + 20 pts) State both Rolle's Theorem and The Mean Value Theorem. Then prove that Rolle's Theorem implies The Mean Value Theorem.

   Ans:

   See page 245-247 of the textbook.

3. (30 pts) Find all critical points of $f(x) = x^{\frac{1}{3}}(x - 4)$. For each one of them, use first derivative test to determine whether it corresponds to a local minimum, a local maximum or neither.

   Ans:

   See page 255 of the textbook.