Brief solutions to Quiz 3

Oct 17, 2023:

1. (20 pts)

   Suppose f is a continuous function on $\mathbb{R}$ and $f(1) < 0$. Show that there exists a $\delta > 0$ such that $f(x) < 0$ on $(1-\delta, 1+\delta)$.

   Ans:

   Take $\varepsilon = \dfrac{-f(1)}{2} > 0$. From continuity of f at 1, there exists a $\delta > 0$ such that $f(x) - f(1) < \dfrac{-f(1)}{2}$, that is $f(x) < f(1) - \dfrac{f(1)}{2} = \dfrac{f(1)}{2} < 0$, on $(1-\delta, 1+\delta)$.

2. (20+20 pts)

   (a) State the Intermediate Value Theorem.

   (b) Use it to show that $x = \cos x$ has at least a solution (take it for granted that both $y = x$ and $y = \cos x$ are continuous functions).

   Ans:

   (a) See Theorem 11 of the textbook for the statement.

   (b) The function $f(x) = x - \cos(x)$ is continuous and satisfies $f(0) < 0 < f(1)$, therefore there exists a $c \in (0,1)$ such that $f(c) = 0$, that is, $c = \cos(c)$.

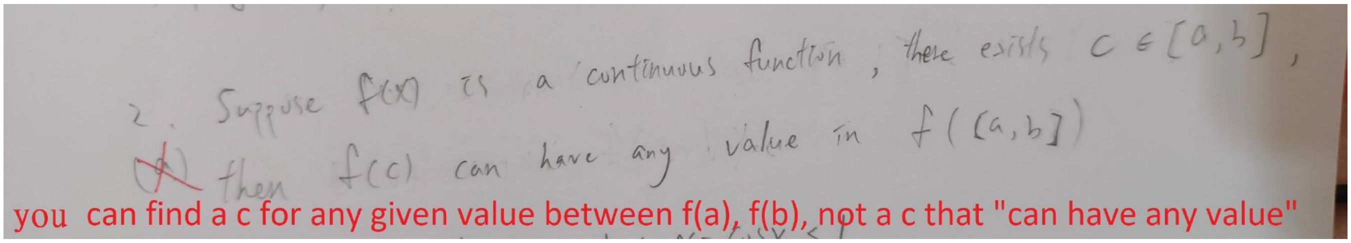

Figure 1: A common mistake to problem 2. See Lecture 04, page 8 for correct interpretation of "f takes any value between $f(a)$ and $f(b)$.

3. (20+20 pts)

(a) State precise definition of $\lim_{x\to 0^+} f(x) = \infty$.

(b) Use it to prove that $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$.

Ans:

(a): For any $B > 0$, there exists a corresponding $\delta > 0$ such that $0 < x < \delta$ implies $f(x) > B$. (See also p131 of the textbook)

(b): Take $\delta = 1/\sqrt{B}$. Then $0 < x < \delta$ implies $f(x) = \frac{1}{x^2} > \frac{1}{\delta^2} = B$.

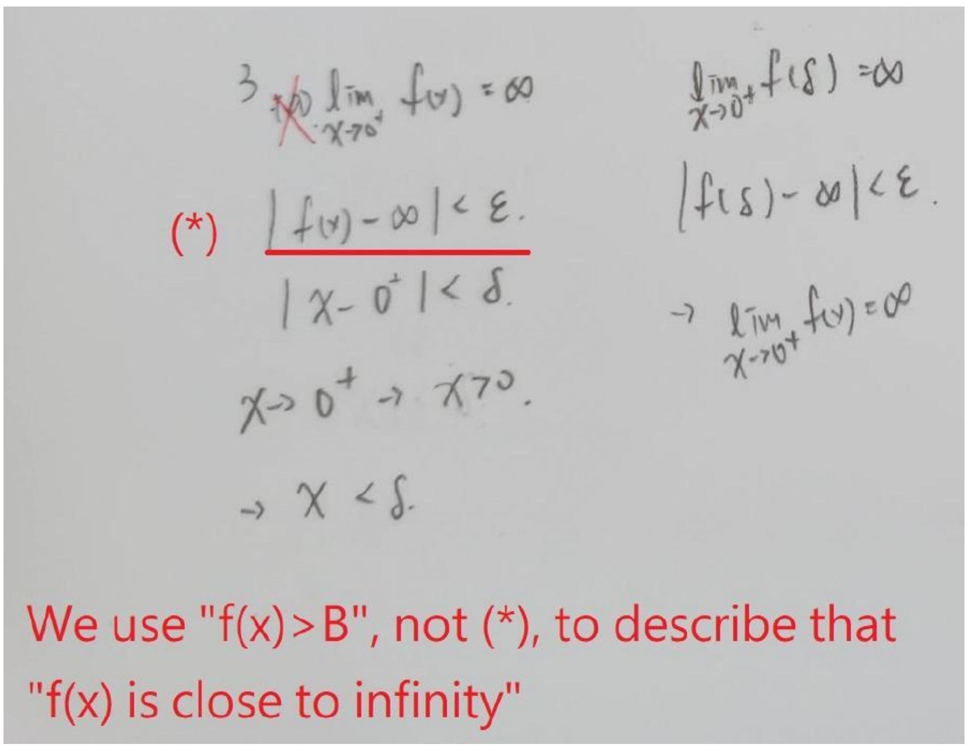

Figure 2: A common mistake to problem 3