Brief solutions to Quiz 2

Sep 26, 2023:

1. (15+20 pts) Give precise definition of $\lim_{x \to c} f(x) = L$ and use it to prove that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

Ans:

See the textbook for definition.

For any $\varepsilon > 0$ (we will modify it to $0 < \varepsilon < \frac{1}{2}$ shortly), we have

$$
\begin{aligned}
& |\frac{1}{x} - \frac{1}{2}| < \varepsilon \\
\Longleftrightarrow \quad & -\varepsilon < \frac{1}{x} - \frac{1}{2} < \varepsilon \\
\Longleftrightarrow \quad & -\varepsilon + \frac{1}{2} < \frac{1}{x} < \varepsilon + \frac{1}{2} \quad \text{(here we need } -\varepsilon + \frac{1}{2} > 0, \text{ or } \varepsilon < \frac{1}{2}, \text{ for next "} \Longleftrightarrow \text{")} \\
\Longleftrightarrow \quad & \frac{1}{\varepsilon + \frac{1}{2}} < x < \frac{1}{-\varepsilon + \frac{1}{2}} \\
\Longleftrightarrow \quad & \frac{1}{\varepsilon + \frac{1}{2}} - 2 < x - 2 < \frac{1}{-\varepsilon + \frac{1}{2}} - 2
\end{aligned}
$$

It is easy to see that if $0 < \varepsilon < \frac{1}{2}$, then $\frac{1}{\varepsilon + \frac{1}{2}} - 2 < 0$ and $\frac{1}{-\varepsilon + \frac{1}{2}} - 2 > 0$. It suffices to take

$$\delta = \min\left\{2 - \frac{1}{\varepsilon + \frac{1}{2}},\ \frac{1}{-\varepsilon + \frac{1}{2}} - 2\right\} > 0. \tag{1}$$

Then

$$
\begin{aligned}
& \frac{1}{\varepsilon + \frac{1}{2}} - 2 < x - 2 < \frac{1}{-\varepsilon + \frac{1}{2}} - 2 \\
\Longleftarrow \quad & -\delta < x - 2 < \delta \\
\Longleftarrow \quad & 0 < |x - 2| < \delta
\end{aligned}
$$

Thus we have found $\delta > 0$ for every $0 < \varepsilon < \frac{1}{2}$ from (1). The *delta* found for ε with $0 < \varepsilon < \frac{1}{2}$ can be used for any $\varepsilon \geq \frac{1}{2}$. The proof is completed.

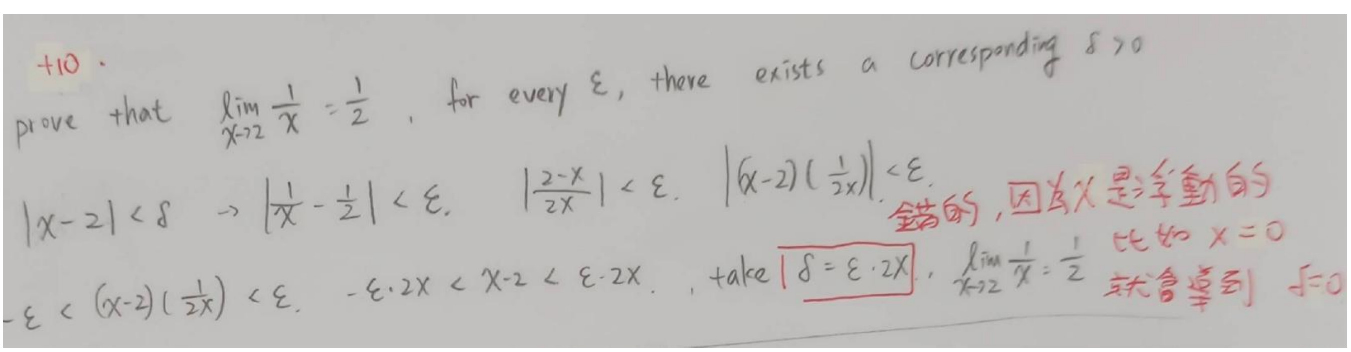

Figure 1: A common mistake to problem 1

2. (15+20 pts) Give precise definition of $\lim_{x \to c^+} f(x) = L$ and use it to prove that:

If $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^+} g(x) = M$, then $\lim_{x \to c^+} \big(3f(x) - 4g(x)\big) = 3L - 4M$.

Ans:

See the textbook for definition.

For any $\varepsilon > 0$, there exist corresponding $\delta_1 > 0$ and $\delta_1 > 0$, such that

$$0 < x - c < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{7}$$

and

$$0 < x - c < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{7}.$$

Take $\delta = \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned}
& 0 < x - c < \delta \\
\implies & \begin{cases} |f(x) - L| < \frac{\varepsilon}{7} \\ |g(x) - M| < \frac{\varepsilon}{7} \end{cases} \\
\implies & \begin{cases} |3f(x) - 3L| < \frac{3\varepsilon}{7} \\ |4g(x) - 4M| < \frac{4\varepsilon}{7} \end{cases} \\
\implies & \begin{cases} \frac{-3\varepsilon}{7} < 3f(x) - 3L < \frac{3\varepsilon}{7} \\ \frac{-4\varepsilon}{7} < 4g(x) - 4M < \frac{4\varepsilon}{7} \end{cases} \\
\implies & \frac{-3\varepsilon}{7} - \frac{4\varepsilon}{7} < (3f(x) - 3L) - (4f(x) - 4L) < \frac{4\varepsilon}{7} - \frac{-3\varepsilon}{7} \\
\implies & -\varepsilon < \big(3f(x) - 4g(x)\big) - (3L - 4M) < \varepsilon \\
\implies & \big|\big(3f(x) - 4g(x)\big) - (3L - 4M)\big| < \varepsilon
\end{aligned}$$

This proves $\lim_{x \to c^+} \big(3f(x) - 4g(x)\big) = 3L - 4M$.

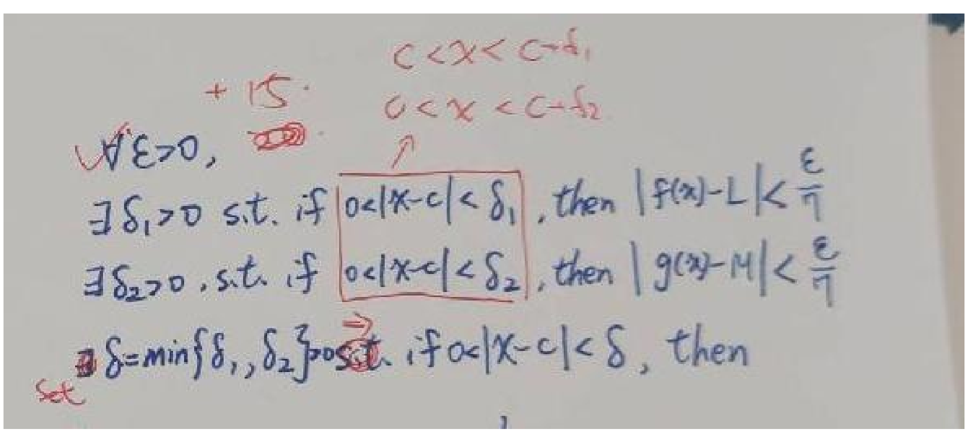

Figure 2: A common mistake to problem 1

3. (30 pts) Evaluate $\lim_{\theta\to 0}\dfrac{\sin(1-\cos\theta)}{\theta^2}$. You can use what you know about $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ without proof.

Ans:

Since $1-\cos\theta = 2\sin^2\dfrac{\theta}{2}$, we have

$$\lim_{\theta\to 0}\frac{\sin(1-\cos\theta)}{\theta^2} = \lim_{\theta\to 0}\left(\left(\frac{\sin\left(2\sin^2\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}}\right)\left(\frac{2\sin^2\frac{\theta}{2}}{2\left(\frac{\theta}{2}\right)^2}\right)\left(\frac{2\left(\frac{\theta}{2}\right)^2}{\theta^2}\right)\right)$$

$$= \lim_{\sin\theta\to 0}\left(\frac{\sin\left(2\sin^2\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}}\right)\cdot\lim_{\theta\to 0}\left(\frac{2\sin^2\frac{\theta}{2}}{2\left(\frac{\theta}{2}\right)^2}\right)\cdot\lim_{\theta\to 0}\left(\frac{2\left(\frac{\theta}{2}\right)^2}{\theta^2}\right) = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}$$

Here we have used the fact $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ (Theorem 7).