Brief solutions to Midterm 02 (63.69/112)

Dec 05, 2023. Show all details to get potential partial credits. 14 pts each problem, total 112 pts

1. (5.94/14) Let g be the inverse function of f, with f and g both twice differentiable (i.e. both first and second derivative exist). Suppose $\begin{matrix} f(1)=2, & f(2)=3, & f(3)=4 \\ f'(1)=5, & f'(2)=6, & f'(3)=7 \\ f''(1)=8, & f''(2)=9, & f''(3)=10 \end{matrix}$.

   Find $g(2)$, $g'(2)$ and $g''(2)$.

   Ans:

$$f(1)=2 \implies g(2)=1.$$

$$g(f(x))=x \implies g'(f(x))\cdot f'(x)=1 \implies g'(f(x))=\frac{1}{f'(x)} \implies g'(2)=\frac{1}{f'(1)}=\frac{1}{5}$$

$$\frac{d}{dx}\Big((g'(f(x))\cdot f'(x)=1\Big) \implies g''(f(x))\cdot(f'(x))^2+g'(f(x))\cdot f''(x)=0$$

$$\implies g''(f(x))=\frac{-g'(f(x))\cdot f''(x)}{(f'(x))^2}=\frac{-f''(x)}{(f'(x))^3} \implies g''(2)=\frac{-f''(1)}{(f'(1))^3}=\frac{-8}{125}$$

2. (11.19/14) Evaluate the derivative of $(\sin^{-1}x)^x + x^{\tan x}$, where $0<x<1$.

   Ans:

$$f(x)^{g(x)}=(e^{\ln f(x)})^{g(x)}=e^{g(x)\ln f(x)}$$

$$\implies \frac{d}{dx}f(x)^{g(x)}=\frac{d}{dx}\big(g(x)\ln f(x)\big)e^{g(x)\ln f(x)}=\Big(g'(x)\ln f(x)+\frac{f'(x)}{f(x)}g(x)\Big)f(x)^{g(x)}$$

$$\text{Answer}=\Big(\ln\sin^{-1}x+\frac{x}{\sqrt{1-x^2}\sin^{-1}x}\Big)(\sin^{-1}x)^x+\Big((\sec^2 x)\ln x+\frac{\tan x}{x}\Big)x^{\tan x}$$

3. (10.25/14) Let $f(x)=\begin{cases} e^{(\frac{-1}{x^2})}, & x\neq 0 \\ 0, & x=0 \end{cases}$. Is f differentiable at $x=0$? Is f twice differentiable at $x=0$?

   Ans:

   Yes. Yes.

   $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{\frac{1}{x}}{e^{x^{-2}}}$ (apply L'Hôpital's Rule for $\frac{\pm\infty}{\infty}$ once) $=\lim\limits_{x\to 0}\frac{-x^{-2}}{-2x^{-3}e^{x^{-2}}}=\lim\limits_{x\to 0}\frac{x}{2e^{x^{-2}}}=0.$

   $f'(x)=\frac{2}{x^3}e^{(\frac{-1}{x^2})}$ for $x\neq 0$.

$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{2x^{-4}}{e^{x^{-2}}}$ (apply L'Hôpital's Rule for $\frac{\pm\infty}{\infty}$ twice) $=$ $\lim_{x\to 0} \frac{-8x^{-5}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} \frac{4x^{-2}}{e^{x^{-2}}} = \lim_{x\to 0} \frac{-8x^{-3}}{-2x^{-3}e^{x^{-2}}} = \lim_{x\to 0} \frac{4}{e^{x^{-2}}} = 0$

4. (9.84/14) Let $f(x)$ be defined for all $x \in \mathbb{R}$ and $L(x) = m(x-a) + f(a)$ for some constants m and a.

   True or false?

   If $f(x) = L(x) + \varepsilon \cdot (x-a)$ with $\lim_{x\to a} \varepsilon = 0$, then f is differentiable at a.

   Ans:

   True.

$$0 = \lim_{x\to a} \varepsilon = \lim_{x\to a} \frac{f(x) - L(x)}{x - a} = \lim_{x\to a} \frac{f(x) - f(a) - m(x-a)}{x - a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - m$$

$$\implies \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = m \implies f'(a) = m$$

5. (9.65/11) Find the limits of the following expressions:

$$\text{(a)} \quad \lim_{x\to 0^+} (1-2x)^{\frac{1}{x}} \qquad \text{(b)} \quad \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\tan x}$$

   (a) **Ans**:

$$= \lim_{x\to 0^+} \left(e^{\ln(1-2x)}\right)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{\ln(1-2x)}{x}} = e^{\left(\lim_{x\to 0^+} \frac{\ln(1-2x)}{x}\right)} = e^{\left(\lim_{x\to 0^+} \frac{\frac{-2}{1-2x}}{1}\right)} = e^{-2}$$

   (b) **Ans**:

$$= \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\tan x} = \lim_{x\to 0} \frac{x}{\sin x}(\cos x)(x \sin\frac{1}{x}) = \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \cos x\right)\left(\lim_{x\to 0} x \sin\frac{1}{x}\right) = 0$$

6. (4.12/14) Let $y(x)$ be the solution of $y'(x) = e^{-x^2} + 1$, $y(0) = -1$. Show that $y(x) = 0$ has exactly one solution.

   Ans:

   First, we show that $y(2) > 0$. Otherwise, if $y(2) \le 0$, then by Mean Value Theorem, there exists a $c \in (0,2)$ such that $y'(c) = \frac{y(2) - y(0)}{2 - 0} \le \frac{0 - (-1)}{2 - 0} = \frac{1}{2}$, a contradiction since $y'(x) = e^{-x^2} + 1 > 1$.

   Secondly, since $y(0) < 0$, $y(2) > 0$, by Intermediate Value Theorem, there exists a $c_1 \in (0,2)$ such that $y(c_1) = 0$.

   Finally, we show that $y(x) = 0$ has only one solution. Suppose not, we will have $c_1 \neq c_2$ such that $y(c_1) = y(c_2) = 0$. From Rolle's Theorem, there exists an $\alpha \in (c_1, c_2)$ such that $y'(\alpha) = 0$, a contradiction since $y'(x) = e^{-x^2} + 1 > 1$.

7. (6.53/14) Evaluate

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{\ln k-\ln n}{k}$$

Ans:

$$=\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{\ln\frac{k}{n}}{\frac{k}{n}n}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=n+1}^{2n}\frac{\ln\frac{k}{n}}{\frac{k}{n}}=\int_1^2\frac{\ln x}{x}dx$$

$$=\int_1^2\ln x\,d\ln|x|=\int_1^2\ln x\,d\ln x=\frac{(\ln x)^2}{2}\Big|_1^2=\frac{(\ln 2)^2}{2}$$

8. (6.16/14) Suppose $f(x)$ satisfies $\displaystyle\int_0^{x^3}2^{-t}f(t)dt=\cos x+C$ for some constant C and all $x\in\mathbb{R}$. Find C and $f(2)$.

Ans:

Evaluate at $x=0$ on both sides:

$$0=\int_0^0 2^{-t}f(t)dt=\cos 0+C \implies C=-1.$$

Take the derivative on both sides:

$$\implies \left(2^{-x^3}f(x^3)\right)\cdot 3x^2=-\sin x$$

Evaluate at $x=2^{\frac{1}{3}}$ on both sides:

$$\implies \left(2^{-2}f(2)\right)\cdot 3\cdot 2^{\frac{2}{3}}=-\sin(2^{\frac{1}{3}}) \implies f(2)=-\frac{4\sin(2^{\frac{1}{3}})}{3\cdot 2^{\frac{2}{3}}}$$