Brief solutions to Midterm 01

Oct 24, 2023. <u>Show all details to get potential partial credits.</u> Total 112 pts (72.42/112).

1. (8.15/6+6 pts) Give precise definition of $\lim_{x\to\infty} f(x) = L_1$ and use it to prove that:

   If $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to\infty} g(x) = L_2$, then $\lim_{x\to\infty} \big(3f(x) - 4g(x)\big) = 3L_1 - 4L_2$.

   Ans:

   See page 119 of the the textbook for definition.

   For any $\varepsilon > 0$, there exist corresponding $M_1 > 0$ and $M_2 > 0$, such that

   $$x > M_1 \implies |f(x) - L_1| < \frac{\varepsilon}{7}$$

   and

   $$x > M_2 \implies |g(x) - L_2| < \frac{\varepsilon}{7}.$$

   Take $M = \max\{M_1, M_2\}$, then

   $$\begin{aligned}
   & x > M \\
   \implies & \begin{cases} |f(x) - L_1| < \frac{\varepsilon}{7} \\ |g(x) - L_2| < \frac{\varepsilon}{7} \end{cases} \\
   \implies & \begin{cases} |3f(x) - 3L_1| < \frac{3\varepsilon}{7} \\ |4g(x) - 4L_2| < \frac{4\varepsilon}{7} \end{cases} \\
   \implies & \begin{cases} \frac{-3\varepsilon}{7} < 3f(x) - 3L_1 < \frac{3\varepsilon}{7} \\ \frac{-4\varepsilon}{7} < 4g(x) - 4L_2 < \frac{4\varepsilon}{7} \end{cases} \\
   \implies & \frac{-3\varepsilon}{7} - \frac{4\varepsilon}{7} < (3f(x) - 3L_1) - (4f(x) - 4L_1) < \frac{4\varepsilon}{7} - \frac{-3\varepsilon}{7} \\
   \implies & -\varepsilon < \big(3f(x) - 4g(x)\big) - (3L_1 - 4L_2) < \varepsilon \\
   \implies & \big|\big(3f(x) - 4g(x)\big) - (3L_1 - 4L_2)\big| < \varepsilon
   \end{aligned}$$

   This proves $\lim_{x\to\infty} \big(3f(x) - 4g(x)\big) = 3L_1 - 4L_2$.

2. (8.97/8+8 pts)

   (a) Evaluate $\lim_{\theta\to 0} \dfrac{\sin(1-\cos\theta)}{\sin^2\theta}$.

   Ans:

   Since $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$, we have

$$\lim_{\theta\to 0}\frac{\sin(1-\cos\theta)}{\sin^2\theta} = \lim_{\theta\to 0}\left(\left(\frac{\sin\left(2\sin^2\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}}\right)\left(\frac{2\sin^2\frac{\theta}{2}}{2\left(\frac{\theta}{2}\right)^2}\right)\left(\frac{2\left(\frac{\theta}{2}\right)^2}{\sin^2\theta}\right)\right)$$

$$= \lim_{\sin\theta\to 0}\left(\frac{\sin\left(2\sin^2\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}}\right)\cdot\lim_{\theta\to 0}\left(\frac{2\sin^2\frac{\theta}{2}}{2\left(\frac{\theta}{2}\right)^2}\right)\cdot\lim_{\theta\to 0}\left(\frac{2\left(\frac{\theta}{2}\right)^2}{\sin^2\theta}\right) = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}$$

Here we have used the fact $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ (Theorem 7).

(b) Evaluate $\lim_{z\to x}\frac{\csc(z^2+1)-\csc(x^2+1)}{e^z-e^x}$.

Ans:

$$\lim_{z\to x}\frac{\csc(z^2+1)-\csc(x^2+1)}{e^z-e^x} = \lim_{z\to x}\frac{\csc(z^2+1)-\csc(x^2+1)}{z-x}\lim_{z\to x}\frac{z-x}{e^z-e^x}$$

$$= \frac{\lim_{z\to x}\frac{\csc(z^2+1)-\csc(x^2+1)}{z-x}}{\lim_{z\to x}\frac{e^z-e^x}{z-x}} = \frac{\frac{d}{dx}\csc(x^2+1)}{\frac{d}{dx}e^x}$$

$$= \frac{-\csc(x^2+1)\cdot\cot(x^2+1)\cdot 2x}{e^x}$$

3. (11.79/8+8 pts)

(a) State the Sandwich Theorem.

(b) State the Intermediate Value Theorem.

Ans:

See the textbook.

4. (11.41/8+8 pts)

True or False? If true, prove it. If false, give a counter example.

(a) If $y = f(x)$ is differentiable at $x = c$ then it is continuous at $x = c$.

(b) If $y = f(x)$ is continuous at $x = c$ then it is differentiable at $x = c$.

Ans:

(a) True. Suppose $f'(c)$ exist.

$$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0.$$

(b) False. Let $f(x) = |x|$ and $c = 0$. Then f is continuous at c but not differentiable at c. ($\because \lim_{x\to 0+}\frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0-}\frac{|x|}{x}$)

5. (6.36/10 pts) Evaluate $\frac{d}{dx}\begin{vmatrix} \sin(x+a) & \cos(x+a) & a \\ \sin(x+b) & \cos(x+b) & b \\ \sin(x+c) & \cos(x+c) & c \end{vmatrix}$, where $a, b, c \in \mathbb{R}$.

Ans:

$= \begin{vmatrix} \cos(x+a) & \cos(x+a) & a \\ \cos(x+b) & \cos(x+b) & b \\ \cos(x+c) & \cos(x+c) & c \end{vmatrix} + \begin{vmatrix} \sin(x+a) & -\sin(x+a) & a \\ \sin(x+b) & -\sin(x+b) & b \\ \sin(x+c) & -\sin(x+c) & c \end{vmatrix} + \begin{vmatrix} \sin(x+a) & \cos(x+a) & 0 \\ \sin(x+b) & \cos(x+b) & 0 \\ \sin(x+c) & \cos(x+c) & 0 \end{vmatrix} = 0.$

The first 2 determinants are zero since they both have identical columns. The third determinant is zero since it has a zero column.

6. (7.63/10 pts) Find the smallest positive integer n such that $\frac{d^n}{dx^n}(x^7 \cos x)\Big|_{x=0}$ is nonzero and find this value. Explain.

Ans: Since

$$\frac{d^n}{dx^n}(x^7 \cos x)\Big|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} x^7\Big|_{x=0} \frac{d^{n-k}}{dx^{n-k}} \cos x\Big|_{x=0}$$

and

$$\frac{d^p}{dx^p} x^7\Big|_{x=0} \neq 0 \iff p = 7, \qquad \frac{d^q}{dx^q} \cos x\Big|_{x=0} \neq 0 \iff q = 0, 2, 4, \cdots,$$

It follows that the smallest $n = 7$ $(p = 7, q = 0)$.

Answer $= \binom{7}{7} \cdot 7! \cdot \cos 0 = 7!$

7. (9.54/8+8 pts) Find the derivative of the following functions:

   (a) $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$

   Ans:

   See the solution for section 3.6, problem 59.

   Useful trick: $\left(\frac{t^2}{t^3 - 4t}\right)^3 = \left(1 - \frac{4}{t}\right)^{-3}$

   (b) $y = \tan^2(\sin^3 t)$

   Ans:

   See the solution for section 3.6, problem 67.

8. (8.79/8+8 pts)

   (a) Find the normal line and tangent line of $x^4 + y^4 = 2$ at $(1, 1)$.

   (b) Find $y''(x)$ at $(1, 1)$.

Ans:

(a)

$$x^4 + y^4 = 2 \implies 4x^3 + 4y^3y' = 0 \implies y' = -\frac{x^3}{y^3} = -1$$

$$\text{tangent line :} \quad \frac{y-1}{x-1} = -1, \quad \text{normal line :} \quad \frac{y-1}{x-1} = 1.$$

(b)

$$4x^3+4y^3y' = 0 \implies 12x^2+12y^2(y')^2+4y^3y'' = 0 \implies y'' = \frac{-3x^2 - 3y^2(y')^2}{y^3} = -6$$