Calculus I, Fall 2023

Brief solutions to Final Exam

Jan 09, 2024.

1. (16 pts) Find the solutions for (a): $2\sqrt{xy}\,\dfrac{dy}{dx}=1,\ y(1)=1,\ x,y>0$ and (b): $x\dfrac{dy}{dx}+3y=\dfrac{\sin x}{x^2},\ y(\pi)=0,\ x>0$, respectively.

   Ans:

   (a): $y(x)=\left(\dfrac{3}{2}x^{\frac{1}{2}}-\dfrac{1}{2}\right)^{\frac{3}{2}}$. (b): $y(x)=-\dfrac{1+\cos x}{x^3}$.

2. (64 pts) Evaluate

   (a) : $\displaystyle\int_0^{\frac{\pi}{3}} x\tan^2 x dx$ (b) : $\displaystyle\int_0^1 \tanh(e^x)e^x\,dx$ (c) : $\displaystyle\int_0^1 \tan^{-1}x\,dx$ (d) : $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\sec x\,dx$

   (e) : $\displaystyle\int\sqrt{\frac{4-x}{x}}\,dx$ (f) : $\displaystyle\int\frac{x^2+2x+1}{(1+x^2)^2}dx$ (g) : $\displaystyle\int_0^1\frac{1}{\sqrt{1+e^x}}\,dx$ (h) : $\displaystyle\int_0^{\frac{\pi}{4}}\frac{1}{1-\sin x}dx$

   Ans:

   (a): $\dfrac{\sqrt{3}}{3}\pi-\ln 2-\dfrac{\pi^2}{18}$. (b): $\ln\left(\dfrac{\cosh(e)}{\cosh(1)}\right)$. (c): $\dfrac{\pi}{4}-\dfrac{\ln 2}{2}$. (d): $\dfrac{\sqrt{2}}{2}-\dfrac{1}{2}\ln(\sqrt{2}+1)$.

   (e): $4\sin^{-1}\dfrac{\sqrt{x}}{2}+\sqrt{4x-x^2}+C$. (f): $\tan^{-1}x-\dfrac{1}{x^2+1}+C$.

   (g): $\ln\left(\dfrac{\sqrt{e+1}-1}{\sqrt{e+1}+1}\right)-\ln\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)$. (h): $\sqrt{2}$.

3. (12 pts) Order e^x, $x^{\ln x}$, $(\ln x)^x$ and x^x from slowest to fastest growing rate as $x\to\infty$. For simplicity of expression, you can use the notations $a\ll b$ to represent b grows faster than a, and $a\approx b$ to represent a and b grow at the same rate. Give details.

   Ans:

   $$x^{\ln x}\ll e^x\ll(\ln x)^x\ll x^x$$

4. (18 pts) Consider the curve $\mathcal{C}=\{x^{\frac{2}{3}}+y^{\frac{2}{3}}=1\}$ and its interior $\mathcal{R}=\{x^{\frac{2}{3}}+y^{\frac{2}{3}}\le 1\}$.

   (a) Find the arc length of $\mathcal{C}$.

   (b) Find the area of the surface obtained by rotating the curve $\mathcal{C}$ around the y-axis.

   (c) Find the volume of the solid obtained by rotating the region $\mathcal{R}$ around the y-axis.

   Ans:

   (a): 6. (b): $\dfrac{12}{5}\pi$. (c): $\dfrac{32}{105}\pi$.