

Brief solutions to selected problems in homework 15

1. Section 8.3: Solutions, common mistakes and corrections:

34

$$\int \sec^3 x \, dx$$

$$= (\sec x) \tan x - \int (\sec x) (\sec^2 x) \, dx$$

$$\int \sec^3 x \, dx = \tan x \sec x - \int \tan^2 x \sec x \, dx$$

$$= \tan x \sec x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \tan x \sec x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2} (\tan x \sec x + \int \sec x \, dx)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

Figure 1: Solution to Section 8.3, problem 34

$$\int \sin^2 \theta \cos^3 \theta \, d\theta$$

$$= \int \frac{(1 - \cos 2\theta)}{2} \cos^3 \theta \, d\theta$$

$$= \frac{1}{2} \left[\int \cos^3 \theta \, d\theta - \int \cos \theta \cos^3 \theta \, d\theta \right]$$

$$= \frac{1}{2} \left[\frac{\sin^3 \theta}{3} - \frac{1}{2} \int (\cos 4\theta + \cos 2\theta) \, d\theta \right]$$

$$= \frac{1}{2} \left[\frac{\sin^3 \theta}{3} - \frac{1}{2} \left[\frac{\sin 4\theta}{4} + \frac{\sin 2\theta}{2} \right] \right] + C$$

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
 $\Rightarrow \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

Figure 2: Solution to Section 8.3, problem 57

2. Section 8.4: Solutions, common mistakes and corrections:

$$\begin{aligned}
 & \int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}} \\
 &= \int_0^{\ln 4} \frac{de^t}{\sqrt{e^{2t} + 3^2}} \quad \left\{ \begin{array}{l} x = e^t \\ dx = e^t dt \end{array} \right. \\
 &= \int_1^4 \frac{dx}{\sqrt{x^2 + 3^2}} \\
 & \quad \left\{ \begin{array}{l} x = 3 \tan \theta \quad \theta = \tan^{-1}(\frac{x}{3}) \\ dx = 3 \sec^2 \theta d\theta \\ \sqrt{x^2 + 3^2} = 3 \sec \theta \end{array} \right. \\
 &= \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \frac{3 \sec^2 \theta d\theta}{3 \sec \theta} = \ln |\tan \theta + \sec \theta| \Big|_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \\
 &= \ln 9 - \ln |1 + \sqrt{10}| \quad \# \\
 & \quad \sqrt{\sec^2 \theta} = \sec \theta
 \end{aligned}$$

Figure 3: Solution to Section 8.4, problem 35

$$\begin{aligned}
 & \int_{1/2}^{1/4} \frac{2 dt}{\sqrt{t} + 4t\sqrt{t}} \\
 &= \int_{\tan^{-1}(\frac{\pi}{6})}^{\tan^{-1}(\frac{\pi}{4})} \frac{2 d2\sqrt{t}}{1 + 4t} \quad \left\{ \begin{array}{l} 2\sqrt{t} = \tan \theta \\ d2\sqrt{t} = \sec^2 \theta d\theta \end{array} \right. \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2 \sec^2 \theta d\theta}{\sec^2 \theta} = 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 2 \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{\pi}{6}
 \end{aligned}$$

Figure 4: Solution to Section 8.4, problem 37

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$$\int \sqrt{\frac{4-x}{x}} dx, \text{ Let } \sqrt{x} = u \Rightarrow dx = 2u du$$

$$= \int \frac{\sqrt{4-u^2}}{u} \cdot (2u du)$$

$$= 2 \int \sqrt{4-u^2} du$$

$$= 2 \int \sqrt{4-4\sin^2\theta} (2\cos\theta d\theta)$$

$$= 8 \int \cos^2\theta \cos\theta d\theta = 8 \int \cos\theta \cos^2\theta d\theta$$

$$= 4 \int \frac{1+\cos 2\theta}{2} d\theta = 4 \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= 4 \left(\theta + \sin\theta \cos\theta \right) + C$$

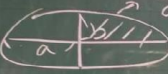
$$= 4 \left(\sin^{-1} \frac{u}{2} + \frac{u}{2} \cos \left(\sin^{-1} \frac{u}{2} \right) \right)$$

$$= 4 \sin^{-1} \frac{u}{2} + 2u \cdot \frac{\sqrt{4-u^2}}{2} + C$$

$$= 4 \sin^{-1} \frac{\sqrt{x}}{2} + \sqrt{x} \sqrt{4-x} + C$$

$$= 4 \sin^{-1} \frac{\sqrt{x}}{2} + \sqrt{4x-x^2} + C$$

Figure 5: Solution to Section 8.4, problem 45

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$


$$A = 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx$$

$$x = a \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$= 4 \int_0^{\frac{\pi}{2}} b \sqrt{\cos^2\theta} \cdot a \cos\theta d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} ab \cos^2\theta d\theta = 4ab \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta$$

$$= 4ab \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2} \right]_0^{\frac{\pi}{2}} = ab\pi$$

Figure 6: Solution to Section 8.4, problem 54

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(a)

$$\int x^3 \sqrt{1-x^2} dx$$

$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} x^2 + \frac{1}{3} \int (1-x^2)^{\frac{3}{2}} \cdot 2x dx$$

$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} x^2 + \frac{1}{3} \cdot \frac{2}{5}(1-x^2)^{\frac{5}{2}}$$

$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} x^2 - \frac{2}{15}(1-x^2)^{\frac{5}{2}} + C$$

(b)

Let $u = 1-x^2 \Rightarrow du = -2x dx$

$$\int x^3 \sqrt{1-x^2} dx$$

$$= -\frac{1}{2} \int x^2 \sqrt{1-x^2} (-2x dx)$$

$$= -\frac{1}{2} \int (1-u) \sqrt{u} du$$

$$= -\frac{1}{2} \int u^{\frac{1}{2}} du + \frac{1}{2} \int u^{\frac{3}{2}} du$$

$$= -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + \frac{1}{2} \cdot \frac{2}{5} u^{\frac{5}{2}} + C$$

Figure 7: Solution to Section 8.4, problem 57(a,b)

57

(a)

$$\int x^3 \sqrt{1-x^2} dx, \quad x = \sin u, \quad 0 \leq u \leq \frac{\pi}{2}$$

$$dx = \cos u du$$

$$= \int \sin^3 u \sqrt{\cos^2 u} \cos u du$$

$$= \int \sin^3 u \cos^2 u du$$

$$= -\int \sin^2 u \cos^2 u d(\cos u)$$

$$= -\int (1 - \cos^2 u) (\cos^2 u) d(\cos u)$$

$$= -\frac{\cos^3 u}{3} + \frac{\cos^5 u}{5} + C$$

Figure 8: Solution to Section 8.4, problem 57(c)

3. Section 8.5: Solutions, common mistakes and corrections:

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$$\int \frac{1}{y^2+1} dy + \int \frac{2y}{(y^2+1)^2} dy$$

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$$\int \frac{y^2+2y+1}{(y^2+1)^2} dy = \tan^{-1} y + \int \frac{d(y^2+1)}{(y^2+1)^2}$$

$$= \tan^{-1} y - \frac{1}{y^2+1} + C$$

$$\frac{y^2+2y+1}{(y^2+1)^2} = \frac{B_1y+C_1}{y^2+1} + \frac{B_2y+C_2}{(y^2+1)^2}$$

$$\Rightarrow (B_1y+C_1)(y^2+1) + B_2y+C_2 = y^2+2y+1$$

$$\Rightarrow \begin{cases} B_1=0 & C_1=1 \\ B_2=2 & C_2=0 \end{cases}$$

Figure 9: Solution to Section 8.5, problem 23

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$$\int \frac{9x^3-3x+1}{x^3-x^2} dx$$

$$\textcircled{1} \frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{2}{x} + \frac{-1}{x^2} + \frac{7}{x-1}$$

$$= \int \left(9 + \frac{2}{x} - \frac{1}{x^2} + \frac{7}{x-1} \right) dx$$

$$= 9x + 2 \ln|x| + \frac{1}{x} + 7 \ln|x-1| + C$$

Figure 10: Solution to Section 8.5, problem 35

(39)

$$\int \frac{e^t}{e^{2t} + 3e^t + 2} dt \quad \begin{array}{l} \text{let } e^t = y \\ e^t dt = dy \end{array}$$

$$= \int \frac{dy}{y^2 + 3y + 2}$$

$$= \int \frac{-1}{(y+2)} + \frac{1}{(y+1)} dy = -\ln|y+2| + \ln|y+1| + C$$

$$= \ln \left| \frac{e^t + 1}{e^t + 2} \right| + C$$

Figure 11: Solution to Section 8.5, problem 39

(45)

$$\int \frac{1}{x^{3/2} - x^{1/2}} dx \quad \begin{array}{l} \sqrt{x} = u \\ \Rightarrow \frac{1}{2} \frac{dx}{\sqrt{x}} = du, x = u^2 \end{array}$$

$$= \int \frac{1}{\sqrt{x} \cdot (x - 1)} dx$$

$$= 2 \int \frac{1}{u^2 - 1} du$$

$$= \int \frac{-1}{u-1} + \frac{1}{u+1} du = \ln|u-1| - \ln|u+1| + C$$

$$= \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$$

Figure 12: Solution to Section 8.5, problem 45

$$\begin{aligned}
 & \int \frac{\sqrt{x+1}}{x} dx, \quad \sqrt{x+1} = u \\
 & \Rightarrow x+1 = u^2 \\
 & \Rightarrow dx = 2u du \\
 & = \int \frac{u}{u^2-1} \cdot 2u du \\
 & = 2 \int \frac{u^2}{u^2-1} du = 2 \int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1} \right) du \\
 & = 2 \int \left(1 + \frac{1}{2} \left(\frac{1}{u-1} - \frac{1}{u+1} \right) \right) du = 2u + \ln \left| \frac{u-1}{u+1} \right| \\
 & = 2\sqrt{x+1} + \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C,
 \end{aligned}$$

Figure 13: Solution to Section 8.5, problem 47

4. Chap 8: Additional and Advanced Exercises. Solutions, common mistakes and corrections:

$$\begin{aligned}
 & \int \frac{dx}{1-\sin x} \\
 & = \int \frac{2}{1+z^2} \frac{dz}{1-\frac{z}{1+z^2}} \\
 & = \int \frac{2}{(1-z)^2} dz \\
 & = \frac{2}{1-z} + C = \frac{2}{1-\tan \frac{x}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 & \tan \frac{x}{2} = z \\
 & \Rightarrow \cos x = \frac{1-z^2}{1+z^2} \\
 & \sin x = \frac{2z}{1+z^2} \\
 & \frac{1}{2} \sec^2 \frac{x}{2} dx = dz \\
 & \Rightarrow (1+z^2) dx = dz \\
 & \Rightarrow dx = \frac{2}{1+z^2} dz
 \end{aligned}$$

Figure 14: Solution to Chap 8: Additional and Advanced Exercises, problem 41

49) $\tan \frac{x}{2} = z$

$$\int \sec x dx = \int \frac{dx}{\cos x} = \int \frac{\frac{z}{1+z^2} dz}{\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{z}{1-z^2} dz = \ln \left| \frac{1+z}{1-z} \right| + C$$

$$= \ln \left| \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} \right| + C$$

上下同乘 $\cos \frac{x}{2} + \sin \frac{x}{2}$

$$= \ln \left| \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right| + C = \ln \left| \frac{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \right| + C$$

$$= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C$$

$$= \ln | \sec x + \tan x | + C$$

Figure 15: Solution to Chap 8: Additional and Advanced Exercises, problem 49