

Brief solutions to selected problems in homework 14

1. Section 7.3: Solutions, common mistakes and corrections:

$$\begin{aligned} k &= \frac{d}{dx} \tan^{-1}(\sinh x) + \frac{d}{dx} C \\ &= \frac{1}{1 + \sinh^2 x} \cdot \cosh x = \frac{1}{\cosh x} \end{aligned}$$

$\cosh^2 x - \sinh^2 x = 1$

Figure 1: Solution to Section 7.3, problem 37(a)

$$\begin{aligned} \text{if } y &= \sin^{-1}(\tanh x) + C \\ \text{then } y' &= \frac{\text{sech}^2 x}{\sqrt{1 - \tanh^2 x}} \\ &= \frac{\text{sech}^2 x}{\text{sech} x} \\ &= \text{sech} x \\ \therefore \int \text{sech} x \, dx &= y \end{aligned}$$

Figure 2: Solution to Section 7.3, problem 37(b)

$$\begin{aligned}
 & \int \tanh\left(\frac{x}{7}\right) dx & \frac{dn}{dx} = \left(\frac{x}{7}\right)' = \frac{1}{7} \\
 & = \int 7 \tanh(n) \frac{1}{7} dx \\
 & = \int 7 \tanh(n) dn \\
 & = 7 \int \tanh(n) dn \\
 & = 7 \int \frac{\sinh(n)}{\cosh(n)} dn = 7 \ln|\cosh(n)| + C \\
 & = 7 \ln\left|\cosh\left(\frac{x}{7}\right)\right| + C
 \end{aligned}$$

Figure 3: Solution to Section 7.3, problem 45

2. Section 7.4: Solutions, common mistakes and corrections:

faster : e, h, d

same : c, f, b, a

slower : x, g

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \frac{\log_3 x}{\ln x} \\
 & = \lim_{x \rightarrow \infty} \frac{\ln x / \ln 3}{\ln x} \\
 & = \frac{1}{\ln 3} \neq 0
 \end{aligned}$$

Figure 4: Solution to Section 7.4, problem 5

$$\lim_{x \rightarrow \infty} \frac{e^{x/2}}{e^x} = \lim_{x \rightarrow \infty} e^{-x/2} = 0$$

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^x}{e^x} = \lim_{x \rightarrow \infty} \left(\frac{\ln x}{e} \right)^x = \infty$$

$$\lim_{x \rightarrow \infty} \frac{x^x}{(\ln x)^x} = \lim_{x \rightarrow \infty} \left(\frac{x}{\ln x} \right)^x = \infty$$

∴ d, a, c, b #

$$\lim_{x \rightarrow \infty} \frac{x}{\ln x}$$

L.H. = $\lim_{x \rightarrow \infty} \frac{1}{1/x} = \infty$

Figure 5: Solution to Section 7.4, problem 7

a. T

$$\frac{\frac{1}{x+3}}{\frac{1}{x}} \leq 1 \text{ as } x \rightarrow \infty$$

b. T

$$\frac{\frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x}} = 1 + \frac{1}{x} \rightarrow 1 \text{ as } x \rightarrow \infty$$

Figure 6: Solution to Section 7.4, problem 10(a,b)

C.F.

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right) = 1 \neq 0$$

d.T.

$$2 + \cos x \leq 3$$

$$\rightarrow \frac{2 + \cos x}{2} \leq \frac{3}{2} \text{ if } x \text{ is sufficiently large.}$$

Figure 7: Solution to Section 7.4, problem 10(c,d)

e)

$$\lim_{x \rightarrow \infty} \frac{e^x + x}{e^x}$$

$$= \lim_{x \rightarrow \infty} \left(1 + \frac{x}{e^x}\right)$$

$$= 1 + \lim_{x \rightarrow \infty} \frac{x}{e^x} \stackrel{\infty/\infty, \text{L.H.}}{\rightarrow}$$

$$\Rightarrow 1 + \lim_{x \rightarrow \infty} \frac{1}{e^x}$$

$$= 1 + 0 = 1 = L$$

$\therefore e^x + x = O(e^x)$ is True.

f)

$$\lim_{x \rightarrow \infty} \frac{x \ln x}{x^2}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{\infty/\infty, \text{L.H.}}{\rightarrow}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{1}{x \cdot 1} = 0$$

$\therefore x \ln x = O(x^2)$ is True.

Figure 8: Solution to Section 7.4, problem 10(e,f)

$$T(g) \quad \frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1 \quad \text{for large } x$$

$$F(h) \quad \lim_{x \rightarrow \infty} \frac{\ln x}{\ln(x^2+1)}$$

$$\begin{aligned} \text{L'H} \quad \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\left(\frac{2x}{x^2+1}\right)} &= \lim_{x \rightarrow \infty} \frac{x^2+1}{2x^2} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2x^2}\right) = \frac{1}{2} \neq 0 \end{aligned}$$

Figure 9: Solution to Section 7.4, problem 10(g,h)

$$n, \sqrt{n} \log_2 n, (\log_2 n)^2$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n} \log_2 n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\log_2 n} = \infty$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n} \log_2 n}{(\log_2 n)^2} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\log_2 n} = \infty$$

rate growth

$$n > \sqrt{n} \log_2 n > (\log_2 n)^2$$

$$\Rightarrow (\log_2 n)^2$$

Figure 10: Solution to Section 7.4, problem 24

3. Section 8.2: Solutions, common mistakes and corrections:

$$\begin{aligned}
\int e^{2x} \cos 3x dx &= \frac{1}{3} \sin 3x e^{2x} - \int \frac{1}{3} \sin 3x \cdot (e^{2x})' dx \\
&= \frac{1}{3} \sin 3x e^{2x} - \frac{2}{3} \int \sin 3x \cdot e^{2x} dx \\
&= \frac{1}{3} \sin 3x e^{2x} - \frac{2}{3} \left(\frac{1}{3} (-\cos 3x) e^{2x} - \int \frac{1}{3} (-\cos 3x) (e^{2x})' dx \right) \\
&= \frac{1}{3} \sin 3x e^{2x} + \frac{2}{9} \cos 3x e^{2x} - \frac{4}{9} \int \cos 3x e^{2x} dx \quad \text{移到左式} \\
\Rightarrow (1 + \frac{4}{9}) \int e^{2x} \cos 3x dx &= \frac{1}{3} \sin 3x e^{2x} + \frac{2}{9} \cos 3x e^{2x} \\
\Rightarrow \int e^{2x} \cos 3x dx &= \frac{3}{13} \sin 3x e^{2x} + \frac{2}{13} \cos 3x e^{2x} + C.
\end{aligned}$$

Figure 11: Solution to Section 8.2, problem 23

$$\begin{aligned}
\int e^{\sqrt{3x+9}} dx & \quad \text{Let } u = \sqrt{3x+9} \\
& \quad dx = \frac{2}{3} \sqrt{3x+9} du \\
&= \int e^u \frac{2}{3} \sqrt{3x+9} du \\
&= \frac{2}{3} \int e^u u du \quad \begin{array}{l} f = u \\ df = du \end{array} \\
&= \frac{2}{3} (u e^u - \int e^u du) \quad dg = e^u du \\
&= \frac{2}{3} (e^u (u-1)) \quad g = e^u \\
&= \frac{2}{3} e^{\sqrt{3x+9}} (\sqrt{3x+9} - 1) + C
\end{aligned}$$

Figure 12: Solution to Section 8.2, problem 25


$$\begin{aligned}
 & \int_0^{\frac{\pi}{3}} x \tan^2 x \, dx \\
 & \left(\begin{array}{l} f = x \\ df = dx \end{array} \right) \quad \begin{array}{l} dg = \tan^2 x \, dx \\ g = \tan x - x \end{array} \\
 & = x(\tan x - x) - \int_0^{\frac{\pi}{3}} \tan x - x \, dx \\
 & = x \tan x - x^2 - \left(-\ln|\cos x| - \frac{1}{2}x^2 \right) \Big|_0^{\frac{\pi}{3}} \\
 & = \frac{\sqrt{3}}{3}\pi - \frac{\pi^2}{18} + \ln \frac{1}{2}
 \end{aligned}$$

Figure 13: Solution to Section 8.2, problem 27

$$\begin{aligned}
 & \int x \tan^{-1} x \, dx \quad x \, dx = d\frac{x^2}{2} \\
 & = \frac{1}{2}x^2 \tan^{-1} x - \int \frac{1}{2}x^2 \cdot \left(\frac{1}{x^2+1} \right) dx \\
 & = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx \\
 & = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\
 & = \frac{1}{2}x^2 \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x \\
 & \quad + C
 \end{aligned}$$

Figure 14: Solution to Section 8.2, problem 51

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C$$

$$\stackrel{?}{=} x \cos^{-1} x - \sqrt{1-x^2} + C$$



$\theta = \cos^{-1} x \Rightarrow \sin \theta = \sqrt{1-x^2}$
 \therefore Yes, they are the same

Figure 15: Solution to Section 8.2, problem 75

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C$$

$$\stackrel{?}{=} x \tan^{-1} x - \ln \sqrt{1+x^2} + C$$

$\sec(\tan^{-1} x)$



$\theta = \tan^{-1} x$
 $\Rightarrow \sec(\tan^{-1} x) = \sec \theta$
 $= \sqrt{1+x^2}$
 $\therefore \ln \sec(\tan^{-1} x) = \ln \sqrt{1+x^2}$

Figure 16: Solution to Section 8.2, problem 76