

### Brief solutions to selected problems in homework 13

1. Section 6.4: Solutions, common mistakes and corrections:

23. 
$$\int 2\pi y ds = \int 2\pi y \sqrt{dx^2 + dy^2} = \int 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$
$$\Rightarrow S = \int_{y=1}^{y=2} 2\pi y \sqrt{(x')^2 + 1} dy$$
$$= \int_{y=1}^{y=2} 2\pi y \sqrt{\left(y^3 - \frac{1}{4y^3}\right)^2 + 1} dy \quad (x' = y^3 - \frac{1}{4y^3})$$
$$= \int_1^2 2\pi y \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2} dy$$
$$= \int_1^2 2\pi y \left[ y^3 + \frac{1}{4y^3} \right] dy \quad (\because y^3 + \frac{1}{4y^3} \geq 0 \text{ if } y \in [1, 2])$$
$$= 2\pi \left( \frac{y^5}{5} - \frac{1}{4y} \right) \Big|_1^2 = \frac{253}{20} \pi$$

Figure 1: Solution to Section 6.4, problem 23

(a)  $x = R \pm \sqrt{R^2 - y^2}$

(b)  $x = R + r \cos t$ ,  $y = r \sin t$

(c)  $(R + \sqrt{R^2 - y^2}) = \pm \frac{1}{2} \frac{2y}{\sqrt{R^2 - y^2}} = \mp \frac{y}{\sqrt{R^2 - y^2}}$

$$= \int_{-r}^r 2\pi(R + \sqrt{R^2 - y^2}) \sqrt{1 + \left(\frac{-y}{\sqrt{R^2 - y^2}}\right)^2} dy$$

$$= \int_{-r}^r 2\pi(R + \sqrt{R^2 - y^2}) \frac{r}{\sqrt{R^2 - y^2}} dy$$

$$= \int_{-r}^r \frac{2\pi r R}{\sqrt{R^2 - y^2}} dy + \int_{-r}^r 2\pi r dy$$

$$= \int_{-r}^r 2\pi(R - \sqrt{R^2 - y^2}) \sqrt{1 + \left(\frac{y}{\sqrt{R^2 - y^2}}\right)^2} dy$$

$$= \int_{-r}^r \frac{2\pi r R}{\sqrt{R^2 - y^2}} dy + \int_{-r}^r 2\pi r dy$$

$$= 2 \int_{-r}^r \frac{2\pi r R}{\sqrt{R^2 - y^2}} dy = 4\pi r R \int_{-r}^r \frac{dy}{\sqrt{R^2 - y^2}}$$

$$= 4\pi r R \left[ \sin^{-1} \left( \frac{y}{R} \right) \right]_{-r}^r$$

$$= 4\pi^2 r R$$

Figure 2: Solution to homework 13, problem 2(a)

(b)  $x = R + r \cos t$

$y = r \sin t$

$$\int_0^{2\pi} 2\pi x(t) \cdot \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$\Rightarrow \int_0^{2\pi} 2\pi(R + r \cos t) r dt$$

$$= \int_0^{2\pi} 2\pi r R dt + \int_0^{2\pi} 2\pi r \cos t dt$$

$$= 2\pi r R (2\pi) + 2\pi r \sin t \Big|_0^{2\pi}$$

$$= 4\pi^2 r R$$

Figure 3: Solution to homework 13, problem 2(b)