

## Brief solutions to selected problems in homework 07

### 1. Section 3.11: Solutions, common mistakes and corrections:

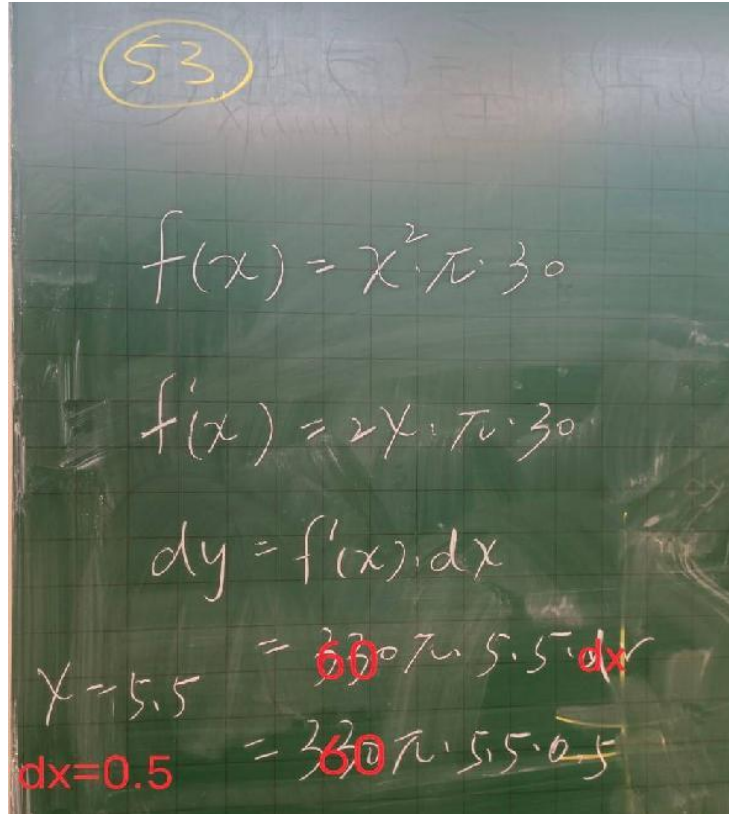


Figure 1: Solution to Section 3.11, problem 53 and some corrections

#### Remark:

" $dy = f'(a)dx$ " is in the notation of "differential" which we skipped.

In the notation of linear approximation, the idea is:

Let  $L(x) = f(a) + f'(a)(x - a)$  be the linear approximation of  $f(x)$  near  $x = a$ ,

$$f(x) \approx L(x) \implies \Delta f \approx \Delta L$$

where  $\Delta f = f(x) - f(a)$  and  $\Delta L = L(x) - L(a)$ . It is easy to check by direct calculating that  $L(x) - L(a) = f'(a)\Delta x$  where  $\Delta x = x - a$ .

Therefore

$$\Delta f \approx \Delta L = f'(a)\Delta x$$

Here  $f(x) = \pi x^2 h = 30\pi x^2$ ,  $a = 5.5$  and  $\Delta x = 0.5$ .

Note:  $dy = f'(a)dx$  is just another way of saying  $\Delta L = f'(a)\Delta x$ .

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1.  $E(a) = 0$   
 $E(a) = f(a) - m(a-a) - C = 0$   
 $\rightarrow f(a) = C$

2°  $\lim_{x \rightarrow a} \frac{f(x) - m(x-a) - C}{x-a} = 0$   
 $\lim_{x \rightarrow a} \left( \frac{f(x) - C}{x-a} - m \right) = 0$   
 $\rightarrow f(a) - m = 0$   
 $m = f'(a)$

3°  $g(x) = m(x-a) + C$   
 $= f'(a)(x-a) + f(a)$




Figure 2: Solution to Section 3.11, problem 66

17(b)  $\sqrt[3]{1.009}$

$|f(x) - L(x)| \leq \frac{1}{2} \left( \max_{c \text{ between } x \text{ and } a} |f''(c)| \right) (x-a)^2$

$f(x) = (1+x)^{1/3}$   
 $L(x) = 1 + \frac{1}{3}x$   
 $a = 0$   
 $x = 0.009$

$\lim_{x \rightarrow a} \frac{(1+x)^{1/3} - (1 + \frac{1}{3}x)}{0.009 - 0} = \frac{-1}{0.009} \times$

$f'(x) = \frac{1}{3}(1+x)^{-2/3}$   
 $f''(x) = -\frac{2}{9}(1+x)^{-5/3}$   
 $0 \leq c \leq 0.009$   
 $\Rightarrow |f''(c)| = \frac{2}{9} |(1+x)^{-5/3}| \leq \frac{2}{9}$

$\Rightarrow |f(0.009) - L(0.009)| \leq \frac{1}{2} \cdot \frac{2}{9} \cdot (0.009 - 0)^2 = 9 \cdot 10^{-6}$

Figure 3: Solution to Homework 07, problem 4

2. Chapter 3, additional and advanced problems: Solutions, common mistakes and corrections:

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$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 = f(0)$$
$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{x} - 0}{x}$$
$$= \lim_{x \rightarrow 0} \left( \frac{1 - \cos x}{x^2} \right) \left( \frac{1 + \cos x}{1 + \cos x} \right)$$
$$= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^2 \left( \frac{1}{1 + \cos x} \right) = \frac{1}{2}$$

$\therefore f(x)$  is ~~continuous~~ diff at 0 ✓

Figure 4: Solution to Chapter 3, additional and advanced problems: problem 16

$$\begin{aligned}
 & 21) f(x)g(x) - f(x_0)g(x_0) \\
 &= f(x)g(x) - f(x_0)g(x) + \underbrace{f(x_0)g(x) - f(x_0)g(x_0)}_{=0 \text{ since } f(x_0)=0} \\
 & \lim_{x \rightarrow x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
 &= \lim_{x \rightarrow x_0} \frac{g(x)[f(x) - f(x_0)]}{x - x_0} + \lim_{x \rightarrow x_0} \underbrace{f(x_0)}_{\neq 0} \underbrace{[g(x) - g(x_0)]}_{\neq 0} \\
 &= \lim_{x \rightarrow x_0} g(x) \cdot f'(x) \neq 0 = g(x_0) \cdot f'(x_0) \quad \because g(x) \text{ is diff}
 \end{aligned}$$

Figure 5: Solution to Chapter 3, additional and advanced problems: problem 21

$$\begin{aligned}
 h(x) &= \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases} \\
 f(x) &= x, \quad f \text{ diff at } x=0, \quad f(0)=0 \\
 g(x) &= \begin{cases} x \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases} \quad g \text{ is contin at } 0 \\
 h &= fg \\
 \therefore & \text{ by 21, } h \text{ is diff at } x=0 \quad \#
 \end{aligned}$$

Figure 6: Solution to Chapter 3, additional and advanced problems: problem 22(d)

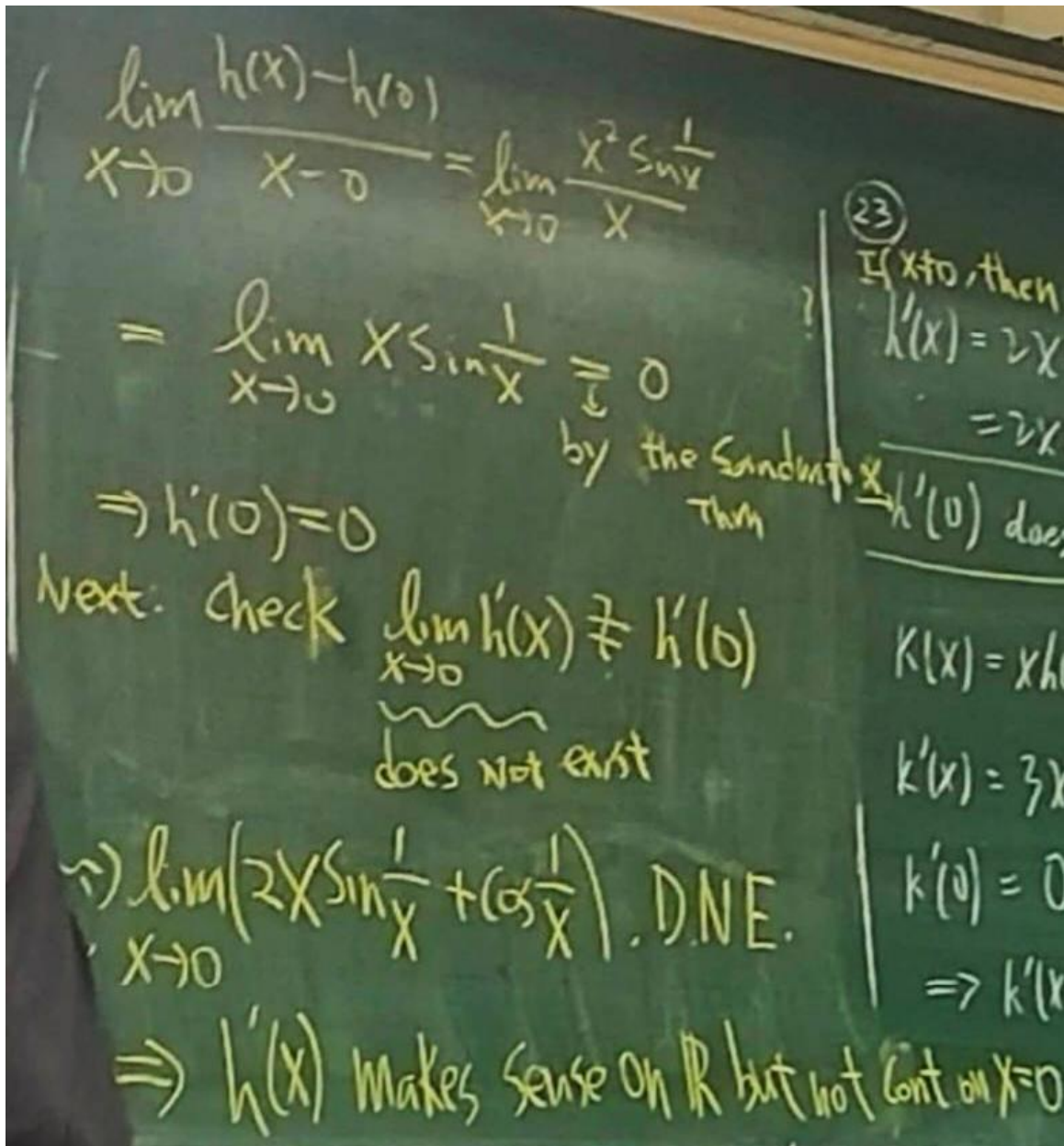


Figure 7: Solution to Chapter 3, additional and advanced problems: problem 23, part 1

$$K(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

① If  $x \neq 0$ , then

$$K'(x) = 3x^2 \sin \frac{1}{x} + x^3 \left( \cos \frac{1}{x} \cdot \left( -\frac{1}{x^2} \right) \right)$$

$$= 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$$

② If  $x = 0$ ,

$$K'(0) = \lim_{x \rightarrow 0} \frac{K(x) - K(0)}{x - 0} = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

Hence,  $K'(x)$  is continuous at  $x=0$   
since  $\lim_{x \rightarrow 0} K'(x) = 0$  as  $x$  tends to 0

Figure 8: Solution to Chapter 3, additional and advanced problems: problem 23, part 2