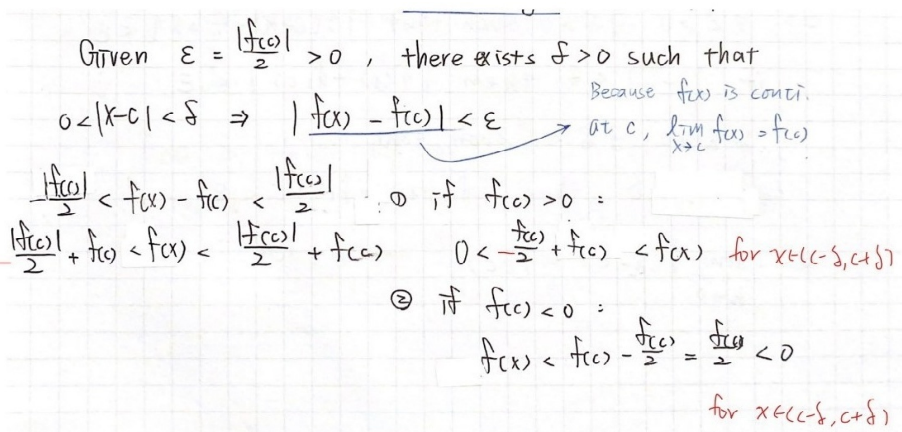
Brief solutions to selected problems in homework 03

1. Section 2.5: Solutions, common mistakes and corrections:



Given $\varepsilon = \frac{|f(c)|}{2} > 0$, there exists $\delta > 0$ such that

$0 < |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$ Because $f(x)$ is conti. at c, $\lim_{x\to c} f(x) = f(c)$

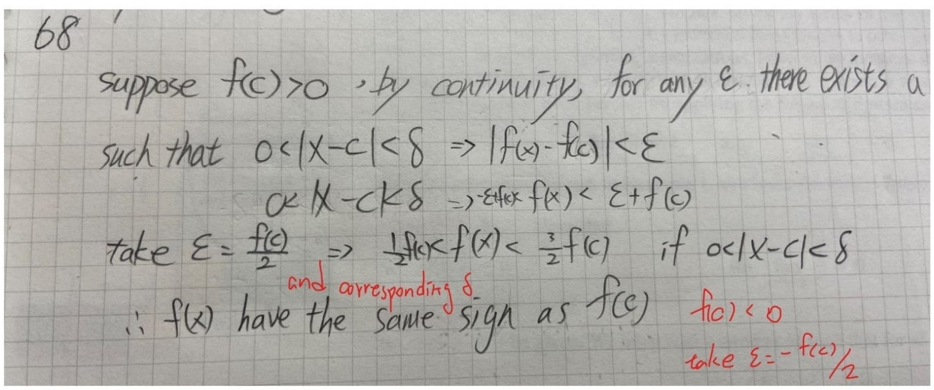
$-\frac{|f(c)|}{2} < f(x) - f(c) < \frac{|f(c)|}{2}$

$-\frac{|f(c)|}{2} + f(c) < f(x) < \frac{|f(c)|}{2} + f(c)$

① if $f(c) > 0$: $0 < -\frac{f(c)}{2} + f(c) < f(x)$ for $x \in (c-\delta, c+\delta)$

② if $f(c) < 0$: $f(x) < f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} < 0$ for $x \in (c-\delta, c+\delta)$

Figure 1: Solution to Section 2.5, problem 68



68

suppose $f(c)>0$, by continuity, for any ε there exists a δ such that $0<|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

$0<|x-c|<\delta \Rightarrow -\varepsilon+f(c) < f(x) < \varepsilon + f(c)$

take $\varepsilon = \frac{f(c)}{2}$ and corresponding δ $\Rightarrow \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c)$ if $0<|x-c|<\delta$

$\therefore$ $f(x)$ have the same sign as $f(c)$

$f(c)<0$ take $\varepsilon = -f(c)/2$

Figure 2: Common mistakes to Section 2.5, problem 68

2. Section 2.6: Solutions, common mistakes and corrections:

Sec 2.6

61. For any $M>0$, take $\delta = \left(\frac{1}{M}\right)^{\frac{3}{2}} > 0$ s.t.

(a) if $0 < x < \delta$ then $f(x) \geq \frac{1}{x^{\frac{2}{3}}} > \frac{1}{\delta^{\frac{2}{3}}} = M$

(b) if $0 < -x < \delta$ then $f(x) \geq \frac{1}{x^{\frac{2}{3}}} > \frac{1}{\delta^{\frac{2}{3}}} = M$

(c) if $0 < x-1 < \delta$ then $f(x) \geq \frac{2}{(x-1)^{\frac{2}{3}}} > \frac{2}{\delta^{\frac{2}{3}}} = 2M > M$

(d) if $0 < 1-x < \delta$ then $f(x) \geq \frac{2}{(x-1)^{\frac{2}{3}}} > \frac{2}{\delta^{\frac{2}{3}}} = 2M > M$

85. $$\lim_{x\to\infty}\left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right) = \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1+\frac{2}{x}}}$$

$$\left(\lim_{x\to\infty}\frac{1}{x} = 0\right) \quad = \frac{5}{2}$$

92. For any $M>0$, take $\delta = \sqrt{\frac{1}{M}} > 0$,

$0 < |x+5| < \delta \Rightarrow 0 < |x+5|^2 < \delta^2 = \frac{1}{M} \Rightarrow \frac{1}{(x+5)^2} > M$

93. (a) $\lim_{x\to c^-} f(x) = \infty \Leftrightarrow$ For any $M>0$, there exists a $\delta>0$, s.t. if $-\delta < x-c < 0$ then $f(x) > M$

(b) $\lim_{x\to c^+} f(x) = -\infty \Leftrightarrow$ For any $-M<0$, there exists a $\delta>0$ s.t. if $0 < x-c < \delta$ then $f(x) < -M$

(c) $\lim_{x\to c^-} f(x) = -\infty \Leftrightarrow$ For any $-M<0$, there exists a $\delta>0$ s.t. if $-\delta < x-c < 0$ then $f(x) < -M$

Figure 3: Solution to selected problems in Section 2.6, part 1

95. For any $-M<0$, take $\delta = \frac{1}{M} > 0$ s.t.

if $-\delta < x < 0$ then $\frac{1}{x} < \frac{1}{-\delta} = -M$

97. For any $M>0$, take $\delta = \frac{1}{M} > 0$ s.t.

if $0 < x-2 < \delta$ then $\frac{1}{x-2} > \frac{1}{\delta} = M$

Figure 4: Solution to selected problems in Section 2.6, part 2

3. Solution to Homework 03, problem 3:

$\lim_{x\to\infty} f(x) = \infty$ $\Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.
if $N<x$ then $f(x)>M$

$\lim_{x\to\infty} f(x) = -\infty$ $\Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.
if $N<x$ then $f(x)<-M$

$\lim_{x\to-\infty} f(x) = \infty$ $\Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.
if $x<-N$ then $f(x)>M$

$\lim_{x\to-\infty} f(x) = -\infty$ $\Leftrightarrow$ for any $M>0$, there exists a $N>0$ s.t.
if $x<-N$ then $f(x)<-M$

Show: $\lim_{x\to\infty} -x^3 = -\infty$

for any $M>0$, there exists a $N=\sqrt[3]{M}>0$ s.t.

if $x>N>0$ then $x^3>N^3=M$

$\Rightarrow -x^3 < -M$

Figure 5: Solution to Homework 03, problem 3