Volume and Surface Area of Revolution (V03)

Notation:

a: axial variable. r: radial variable. (a,r) could be (x,y) or (y,x).
The generating region:

$$\mathcal{R} = \{(a,r),\ \alpha < a < \beta,\ 0 \le r_1(a) < r < r_2(a)\}$$
$$= \{(a,r),\ 0 \le \gamma < r < \delta,\ a_1(r) < a < a_2(r)\}$$

Volume of revolution generated by region $\mathcal{R}$:

Method of disks ($r_1 = 0$) or washers ($r_1 > 0$):

$$V = \int_\alpha^\beta \pi\big(r_2^2(a) - r_1^2(a)\big)da$$

Method of cylindrical shells:

$$V = \int_\gamma^\delta 2\pi r\big(a_2(r) - a_1(r)\big)dr$$

Arclength of curve $\mathcal{C}$ and area of surface of revolution generated by $\mathcal{C}$:

General formula: $L = \int_*^* ds$, $ds = \sqrt{dx^2 + dy^2}$.

$\mathcal{S} = \int_*^* 2\pi r ds$, where $r = x$ if rotated around y-axis, $r = y$ if rotated around x-axis.

Case 1: If $\mathcal{C} = \{y = f(x) \ge 0,\ a \le x \le b\}$, then $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}\,dx$.

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}dx,$$

Surface area (rotate $\mathcal{C}$ around x-axis):

$$S_{x-\text{axis}} = \int_a^b 2\pi r ds = \int_a^b 2\pi y ds = \int_a^b 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}dx$$

Case 2: If $\mathcal{C} = \{x = g(y) \ge 0,\ c \le y \le d\}$, then $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dx}{dy})^2}\,dy$.

$$L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}dy = \int_c^d \sqrt{1 + \big(g'(y)\big)^2}dy,$$

Surface area (rotate $\mathcal{C}$ around y-axis):

$$S_{y-\text{axis}} = \int_c^d 2\pi r ds = \int_c^d 2\pi x ds = \int_c^d 2\pi g(y)\sqrt{1 + \big(g'(y)\big)^2}dy$$

Case 3: If $\mathcal{C} = \{x = X(t),\, y = Y(t),\, \alpha \le t \le \beta\}$, then $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

$$S_{x-\text{axis}} = \int_\alpha^\beta 2\pi r ds = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi Y(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

$$S_{y-\text{axis}} = \int_\alpha^\beta 2\pi r ds = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi X(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$