Remarks on inverse functions with new notations

The purpose of this document is to derive the formula for derivative of inverse functions. Instead of the notation (from the textbook)

$$\text{original function: } y = f(x), \qquad \text{inverse function: } y = f^{-1}(x),$$

we shall use the following notations

$$\text{original function: } y = f(x), \qquad \text{inverse function: } \underline{x = f^{-1}(y)},$$

which is, in my opining, the better one.

Inverse Function of $y = f(x)$

A necessary and sufficient condition for

$$f : D_f \longmapsto R_f \qquad (f \text{ maps from domain of } f \text{ to range of } f)$$

to have an inverse function is

"f is one-to-one and onto from domain of f to range of f"

If this is the case, we can define the inverse function

$$f^{-1} : R_f \longmapsto D_f \qquad (f^{-1} \text{ maps from range of } f \text{ to domain of } f)$$

Proposition 1 *If the inverse functions of f exists, then*

- $f^{-1}(f(x)) = x$, *for all* $x \in D_f$.
- $f(f^{-1}(y)) = y$, *for all* $y \in R_f$.

Notice that we have deliberately used a different notation (y) for the argument of f^{-1} to avoid possible confusion. This is different from the textbook.

It is better to use different letters (x and y) for elements in D_f and R_f. We will follow this notation through rest of this note.

The inverse function of $y = f(x)$ is thus denoted by $x = f^{-1}(y)$.

The exponential functions are one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}^+$. The inverse function, denote by $\log_a$ maps from $\mathbb{R}^+$ to $\mathbb{R}$. Therefore

Proposition 2 *We have*

$$\log_a(a^x) = x, \text{ for all } x \in \mathbb{R}, \qquad a^{\log_a y} = y, \text{ for all } y \in \mathbb{R}^+.$$

In particular,

$$\ln(e^x) = x, \text{ for all } x \in \mathbb{R}, \qquad e^{\ln y} = y, \text{ for all } y \in \mathbb{R}^+.$$

Derivative of Inverse Functions and Logarithmic Functions

Since

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in D_f,$$

we take the $x-$ derivative on both sides and use the chain rule to get

$$\frac{d}{dy} f^{-1}(f(x)) \cdot \frac{df(x)}{dx} = \frac{d}{dx} x = 1$$

In other words,

$$\frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} \cdot \left(\frac{df(x)}{dx}\right) = \frac{d}{dx} x = 1$$

that is,

$$\frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}} \tag{1}$$

or just simply

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{df(x)}{dx}} \tag{2}$$

and keep in mind that x and y in (2) are related to each other by $y = f(x)$ or $x = f^{-1}(y)$. Evaluating both sides of (2) at x gives (1). Evaluating them on y (so $x = f^{-1}(y)$) gives:

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{df(x)}{dx}\Big|_{x=f^{-1}(y)}} \tag{3}$$

For example, if $y = f(x) = e^x$, then $f^{-1}(y) = \ln y$ and we have

$$\frac{d}{dy} \ln y\Big|_{y=e^x} = \frac{1}{\frac{d}{dx} e^x} = \frac{1}{e^x} = \frac{1}{y} \qquad y > 0.$$

Note that the arguments y (of f^{-1}) and x (of f) are evaluated on different points: one on $f(x)$ and the other on x.

The following is WRONG due to confusion from bad notation:

$$\frac{d}{dx} \ln x = \frac{1}{\frac{d}{dx} e^x} = \frac{1}{e^x} = e^{-x}$$

Example 1 *Let* $f(x) = x^3 - 3x^2 - 1, x \geq 2$. *Find the value of* $\frac{df^{-1}(x)}{dx}$ *at* $x = -1 = f(3)$.

Hint: to avoid confusion, it is better to change the problem to "Find the value of $\frac{df^{-1}(y)}{dy}$ at $y = -1 = f(3)$".