Eg2 $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$

Sol $\deg(x^2+1) < \deg (x-1)(x-2)(x-3)$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-3)}$$

$$\Rightarrow (x^2+1) = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2).$$

$x \leftarrow 1 \Rightarrow 2 = 2A_1$

$x \leftarrow 2 \Rightarrow 5 = -A_2$

$x \leftarrow 3 \Rightarrow 10 = 2A_3$

$\therefore$ Ans $= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$

Eg3 $\displaystyle\int \frac{x^4}{(x^2+1)^2(x-1)}\,dx$

Sol:

$$\frac{x^4}{(x^2+1)^2(x-1)} = \frac{A}{x-1} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$\Rightarrow x^4 = A(x^2+1)^2 + (B_1x+C_1)(x^2+1)(x-1) + (B_2x+C_2)(x-1)$$

$x \leftarrow 1 \Rightarrow A = \frac{1}{4}$

$$\Rightarrow \frac{1}{4}(3x^4-2x^2-1) = (x-1)\left(\begin{array}{c}(B_1x+C_1)(x^2+1)\\ +B_2x+C_2\end{array}\right)$$

$$\Rightarrow \frac{1}{\cancel{4}}(3x^3+3x^2+x+1) = 4\left(\begin{array}{c}\underbrace{(B_1x+C_1)}_{商}(x^2+1)\\ +\underbrace{B_2x+C_2}_{餘}\end{array}\right)$$

$$\begin{array}{r|rrrr} & & & \cancel{3}/4 & \cancel{3}/4 \\ 404 & 3 & 3 & 1 & 1 \\ & 3 & 0 & 3 & \\ \hline & & 3 & -2 & 1 \\ & & 3 & 0 & 3 \end{array}$$

$$= (3x+3)\cdot 4(x^2+1) + (-2x-2)$$

$$\therefore\ 3x^3+3x^2+x+1 = 4\left(\begin{array}{l}(\frac{3}{4}x+\frac{3}{4})(x^2+1)\\ -\frac{1}{2}(x+1)\end{array}\right)$$

$$\Rightarrow B_1x+C_1 = \frac{3}{4}x+\frac{3}{4}$$

$$B_2x+C_2 = -\frac{1}{2}(x+1)$$

$$\text{Ans} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\underbrace{\int\frac{x}{x^2+1}dx}_{①} + \frac{3}{4}\underbrace{\int\frac{1}{x^2+1}dx}_{②}$$

$$-\frac{1}{2}\underbrace{\int\frac{x}{(x^2+1)^2}dx}_{③} - \frac{1}{2}\underbrace{\int\frac{1}{(x^2+1)^2}dx}_{④}$$

$$① = \frac{3}{8}\ln(1+x^2)+C$$

$$② = \frac{3}{4}\tan^{-1}x+C$$

$$③ = -\frac{1}{4}\int\frac{d(x^2+1)}{(x^2+1)^2} = \frac{1}{4}(x^2+1)^{-1}+C$$

④ $x=\tan\theta$

$dx=\sec^2\theta\, d\theta$, $④ = -\frac{1}{2}\int\frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \frac{-1}{2}\int\cos^2\theta\, d\theta$

Eg4 $\int \frac{1}{(x^2+1)(x-1)^2}\,dx$

Sol: $\frac{1}{(x^2+1)(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 1 = A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+C)(x-1)^2$

$x \leftarrow 1 \Rightarrow A_2 = \frac{1}{2}$

$-\frac{1}{2}x^2 + \frac{1}{2} = A_1(x-1)(x^2+1) + (Bx+C)(x-1)^2$

$\Rightarrow -\frac{1}{2}(x+1) = A_1(x^2+1) + (Bx+C)(x-1)$

$x \leftarrow 1 \Rightarrow -1 = 2A_1,\ A_1 = \frac{-1}{2}$

$\Rightarrow \frac{1}{2}x^2 - \frac{1}{2}x = (Bx+C)(x-1) \Rightarrow B = \frac{1}{2},\ C = 0$

Ans $= \frac{-1}{2}\ln|x-1| - \frac{1}{2}(x-1)^{-1} + \frac{1}{4}\ln(1+x^2) + C$

Half-Angle substitution (p547-p548)

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}\, dx, \quad P, Q: \text{polynomials}$$

Let $t = \tan\frac{x}{2}$ $\quad dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\text{Ans} = \int \frac{P\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)}{Q\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)} \frac{2}{1+t^2}\, dt$$

$$= \int \frac{P_1(t)}{Q_1(t)}\, dt \quad : \text{partial fraction}$$

(Replace $t = \tan\frac{x}{2}$ in the end)