$t = \tan x, \; e = \sec x$

Eg1 $\int t^3 e^2 dx = \int t^3 dt$

$\text{or} = \int t^2 e \, de = \int (e^2 - 1) e \, de$

$\text{or} = \int \frac{s^3}{c^5} dx = -\int \frac{(1-c^2)\,dc}{c^5}$

Eg2 $\int t^2 e \, dx$

$= \int t \, de = e \cdot t - \int e \, dt$

$= et - \int e^3 dx = et - \int (t^2+1) e \, dx$

$= et - \int t^2 e \, dx - \int e \, dx$

$\Rightarrow \int t^2 e = \frac{et}{2} - \frac{1}{2} \int \frac{1}{\cos x} dx = \cdots$

Eg3 $\int e^6\,dx$

$= \int e^4\,dt = \int (t^2+1)^2\,dt$

$= \int (t^4+2t^2+1)\,dt$

Eg4 $\int t^4\,dx$

$= \int t^2(e^2-1)\,dx = \int t^2\,dt - \int t^2\,dx$

$= \int t^2\,dt - \int (e^2-1)\,dx$

$= \int t^2\,dt - \int dt - \int dx = \cdots$

Eg5 $\int t^5\,dt \rightarrow \int t^3\,dt \rightarrow \int t\,dt$

or $= \int \frac{s^5}{c^5}\,dx = -\int \frac{s^4\,dc}{c^5} = -\int \frac{(1-c^2)^2}{c^5}\,dc$

# Trigonometric Substitution

Designed for $\int f(x)\,dx$,
where $f(x)$ contains factors
of $a^2+x^2$, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$.
$(a>0)$

$a^2+x^2$: $x = a\tan\theta$, $\quad a^2+x^2 = a^2\sec^2\theta$, $\quad dx = a\sec^2\theta\, d\theta$

$\sqrt{a^2-x^2}$: $x = a\sin\theta$, $\quad a^2-x^2 = a^2\cos^2\theta$, $\quad dx = a\cos\theta\, d\theta$

$\sqrt{x^2-a^2}$: $x = a\sec\theta$, $\quad x^2-a^2 = a^2\tan^2\theta$, $\quad dx = a\tan\theta\sec\theta\, d\theta$

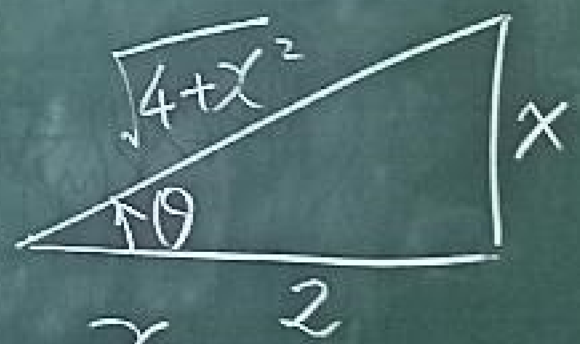
Eg 1 $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$ $\quad x = 2\tan\theta,\ dx = \sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$= \int \frac{2\sec^2\theta}{2\sqrt{\sec^2\theta}}\, d\theta = \int \sec\theta\, d\theta$$

$\sqrt{\sec^2\theta} = \sec\theta$ if $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$= \frac{1}{2}\ln \frac{1+\sin\theta}{1-\sin\theta} + C$$




$\tan\theta = \frac{x}{2} \Longrightarrow \sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{2}\ln\left(\frac{x+\sqrt{4+x^2}}{-x+\sqrt{4+x^2}}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{(x+\sqrt{\ })(x+\sqrt{\ })}{(-x+\sqrt{\ })(x+\sqrt{\ })}\right) + C = \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C$$

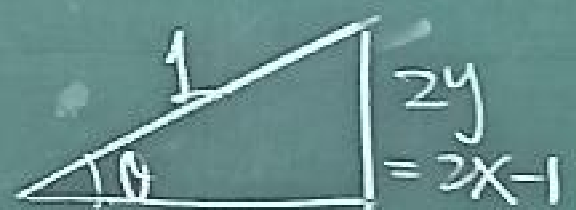
Eg2 $\int \sqrt{x-x^2}\,dx$

Sol $x-x^2$

$= -(x^2-x) = -(x^2-x+\frac{1}{4}-\frac{1}{4})$

$= \frac{1}{4}-(x-\frac{1}{2})^2 = (\frac{1}{2})^2 - y^2$

$y = x-\frac{1}{2} \qquad dx = dy$




$y = \frac{1}{2}\sin\theta,\ dy = \frac{1}{2}\cos\theta\, d\theta$

$\theta = \sin^{-1}(2y) \in (\frac{-\pi}{2}, \frac{\pi}{2})$

$= \int \sqrt{(\frac{\cos\theta}{2})^2}\,\cos\theta\, d\theta \quad (\sqrt{\cos^2\theta} = \cos\theta)$

$= \int \frac{\cos^2\theta}{2}\, d\theta = \int \frac{1+\cos 2\theta}{4}\, d\theta = \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C$

$= \frac{\sin^{-1}(2x-1)}{4} + \frac{1}{4}\underbrace{(2x-1)}_{\sin\theta}\underbrace{(\sqrt{1-(2x-1)^2})}_{\cos\theta} + C$

8.5 Partial fraction for $\int \frac{f(x)}{g(x)}dx$

where $f(x)$ and $g(x)$ are polynomials

Step 1. Make sure $\deg f < \deg g$

If not, $\frac{f}{g} = P + \frac{\tilde{f}}{g}$, $\deg \tilde{f} < \deg g$

Step 2 Find Prime factors of $g$

$(x-r_i)^{m_i}$, $[(x-a_j)^2+b_j^2]^{n_j}$, $i, j = 1, 2, \dots$

Eg: $g(x) = (x-1)\cdot(x-2)^3\cdot(x-5)^2\cdot(x^2+1)(x^2+3)^5$

Step 3. $(x-r)^m$ in $g \Rightarrow \frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots \frac{A_m}{(x-r)^m}$ in $\frac{f}{g}$

$((x-a)^2+b^2)^n$ in $g \Rightarrow \frac{B_1x+C_1}{(x-a)^2+b^2} + \cdots + \frac{B_nx+C_n}{((x-a)^2+b^2)^n}$ in $\frac{f}{g}$

Eg 1 $\displaystyle\int \frac{x^4}{(x-1)^3}\,dx$

Sol $\displaystyle\frac{x^4}{(x-1)^3} = (x+3) + \frac{6x^2-8x+3}{(x-1)^3}$

$$\frac{6x^2-8x+3}{(x-1)^3} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3}$$

Find $A_1, A_2, A_3$

$\Rightarrow 6x^2-8x+3 = A_1(x-1)^2 + A_2(x-1) + A_3$

$x \leftarrow 1 \Rightarrow A_3 = 1$; $\left.\frac{d}{dx}\right|_{x\leftarrow 1} \Rightarrow A_2 = 4$;

$\left.\frac{d^2}{dx^2}\right|_{x\leftarrow 1} \Longrightarrow A_1 = 6$

$\therefore$ Ans $= \frac{x^2}{2} + 3x + 6\ln|x-1| - \frac{4}{(x-1)} - \frac{1}{2(x-1)^2} + C$