

Recall $(C = \cos x, S = \sin x)$
 $(C_2 = \cos 2x, \text{ etc.})$

$m, n, k, l > 0$

$$\int C^m S^{2l+1} dx = - \int C^m (1-C^2)^l dC$$

$$\int \frac{S^{2l+1}}{C^m} dx = - \int \frac{(1-C^2)^l}{C^m} dC$$

$$\int C^{2k+1} S^n dx = \int (1-S^2)^k S^n dS$$

$$\int \frac{C^{2k+1}}{S^n} dx = \int \frac{(1-S^2)^k}{S^n} dS$$

Rm $\int \frac{C^m}{S^{2l+1}} dx = \int \frac{C^m S dx}{S^{2l+2}}$

$$= - \int \frac{C^m}{(1-C^2)^{l+1}} dC \quad \left(\begin{array}{l} \text{Section 8.5} \\ \text{partial fractions} \end{array} \right)$$

Similarly for $\int \frac{S^n}{C^{2k+1}} dx$

If both m, n are even

$$\int C^m S^n dx = \frac{C^{m+1} S^{n-1}}{m+n} + \frac{n-1}{m+n} \int C^m S^{n-2} dx$$

$$\left(\begin{array}{l} \text{Table} \\ \text{in textbook} \end{array} \right) = \frac{C^{m-1} S^{n+1}}{m+n} + \frac{m-1}{m+n} \int C^{m-2} S^n dx$$

Important identities

$$(i) C_2 = 2C^2 - 1 = 1 - 2S^2$$

$$\Rightarrow 1 + C_2 = 2C^2, \quad 1 - C_2 = 2S^2$$

$$(ii) \int S_m S_n dx, \quad \int S_m C_n dx, \quad \int C_m C_n dx$$

$$\text{Use } 2S_m S_n = C_{m-n} - C_{m+n}$$

$$2S_m C_n = S_{m-n} + S_{m+n}$$

$$2C_m C_n = C_{m-n} + C_{m+n}$$

Remark

$$(i) S_2 = 25C$$

$$(ii) \sqrt{1 + \sin x} = ?$$

$$\begin{cases} \sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos x \\ \cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin x \end{cases}$$

$$= \sqrt{1 - \cos\left(x - \frac{\pi}{2}\right)}$$

$$= \sqrt{2 \sin^2\left(\frac{x}{2} - \frac{\pi}{4}\right)}$$

$$= \pm \sqrt{2} \sin\left(\frac{x}{2} - \frac{\pi}{4}\right)$$

Similarly

$$\sqrt{1 - \sin x} = \sqrt{1 + \cos\left(x - \frac{\pi}{2}\right)}$$

$$= \pm \sqrt{2} \cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$$

$$\text{Ex 1 } \int S^2 C^4 dx = ?$$

(Method I) $m=4, n=2$

$$= \frac{C^3 S^3}{4+2} + \frac{4-1}{4+2} \int C^2 S^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{2} \cdot \frac{1}{4} \int (2CS)^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8} \int S_2^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8} \int \frac{1-C_4}{2} dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{16} \left(x - \frac{S_4}{4} \right) + \tilde{C}$$

$$= \frac{\cos^3 x \sin^3 x}{6} + \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \tilde{C}$$

Method II

$$C^4 = \left(\frac{1+C_2}{2}\right)^2 = \frac{1}{4} \left(1 + 2C_2 + \frac{1+C_4}{2}\right)$$

$$S^2 = \frac{1-C_2}{2}$$

$$C^4 S^2 = \frac{1}{16} (1-C_2) (3+4C_2+C_4)$$

$$= \frac{1}{16} \left(\begin{array}{l} 3 + 4C_2 + C_4 \\ - 3C_2 - 4C_2^2 - C_2C_4 \end{array} \right)$$

$$= \frac{1}{16} \left(3 + C_2 + C_4 - 2(1+C_4) - \frac{1}{2}(C_2+C_6) \right)$$

$$= \frac{1}{16} \left(1 + \frac{C_2}{2} - C_4 - \frac{C_6}{2} \right)$$

$$\text{Ans} = \frac{1}{16} \left(x + \frac{\sin 2x}{4} - \frac{\sin 4x}{4} - \frac{\sin 6x}{12} \right) + C$$

Rm: After reduction

$$\begin{matrix} (m, 1) \\ = (2, 0) \end{matrix} \int C^2 dx = \int \frac{1+C_2}{2} dx$$

$$(0, 2) \int S^2 dx = \int \frac{1-C_2}{2} dx$$

$$(2, 2) \int C^2 S^2 dx = \frac{1}{4} \int S_2^2 dx$$

$$= \frac{1}{4} \int \frac{1-C_4}{2} dx$$

$$\int t^m e^n dx, \quad t = \tan x, \quad e = \sec x$$

$$dt = e^2 dx, \quad de = t e dx$$

$$(i) \int t^{2k+1} e^n dx = \int t^{2k} e^{n-1} (t e dx) \\ = \int (e^2 - 1)^k e^{n-1} de$$

$$R_m: \int t^{2k+1} e^n dx = \int \frac{S^{2k+1}}{C^{2k+1+n}} dx = - \int \frac{(1-c^2)^k}{C^{2k+1+n}} dc$$

$$(ii) \int t^m e^{2l} dx = \int t^m e^{2l-2} (e^2 dx) \\ = \int t^m (t^2 + 1)^{l-1} dt \quad (m \text{ can be even or odd})$$

$$\text{However } \int t^m e^{2l} dx = \int \frac{S^m}{C^{m+2l}} dx$$

no progress

~~not easy~~ if $m = \text{even}$

$$(iii) \int t^{2k+1} dx = \int \frac{t^{2k+1}}{e} dx = \int \frac{(e^z - 1)^k}{e} de$$

$$\int \frac{c^{2k+1}}{c^{2k+1}} dx = - \int \frac{(1-c^2)^k}{c^{2k+1}} dc$$

$$(iv) \int t^{2k} dx = \int \frac{t^{2k}}{e^z} dx = \int \frac{t^{2k} dt}{1+t^2}$$

$$= \int \left(\text{Polynomial of } t^2 + \frac{\hat{C}}{1+t^2} \right) dt$$

$$= \int P(t^2) dt + \hat{C} \tan^{-1}(\tan x) + \tilde{C}$$

$$\alpha = \int t^{2k-2} (e^z - 1) dx = \int t^{2k-2} (e^z dx) - \int t^{2k-2} dx$$

$$= \int t^{2k-2} dt - \int t^{2k-2} dx \quad \left(\text{reduce from } \int t^{2k} dx \text{ to } \int t^{2k-2} dx \right)$$

$$(2k, 2l+1) \begin{cases} \nearrow (2k, 2l-1) \\ \searrow (2k-2, 2l+1) \end{cases} \rightarrow \dots \rightarrow (2, 1) \rightarrow (0, 1)$$

$$\int t^2 e^t dx = \int \frac{s^2}{c^3} dx = \int \frac{s^2 c dx}{c^4} = \int \frac{s^2 ds}{(1-s^2)^2} \quad (8.5)$$

$$\downarrow$$

$$\int e dx = \int \frac{1}{c} dx = \int \frac{c dx}{c^2} = \int \frac{ds}{1-s^2}$$

$$= \frac{1}{2} \int \left(\frac{1}{s+1} - \frac{1}{s-1} \right) dx$$

$$= \frac{1}{2} \ln \left| \frac{1+\sin x}{\sin x-1} \right| + \tilde{C}$$

$$= \ln \sqrt{\frac{1+\sin x}{1-\sin x}} + \tilde{C}$$

$$\text{or } \int e dx = \int \frac{e(t+e)}{t+e} = \int \frac{d(t+e)}{t+e}$$

$$= \ln |\tan x + \sec x| + \tilde{C}$$