8. Techniques of integration

8.2 Integration by parts.

$$\int f(x)\,g'(x)\,dx = \int f(x)\,dg(x) = ?$$

Ans: $\frac{d}{dx}\int f g'\,dx = f g' = (fg)' - f'g$

$$= \frac{d}{dx}\left(fg - \int f'g\,dx\right)$$

$$\Rightarrow \int f g'\,dx = fg - \int f'g\,dx$$

or $\int f\,dg = fg - \int g\,df$

Useful if $\int g\,df = \int g f'\,dx$ is easy to compute.

Eg1 $\int \overset{g'}{\underset{f}{x}} \, \overset{f}{\underset{g'}{\cos x}} \, dx$

$$= \begin{cases} \int x \, d\sin x = x\sin x - \int \sin x \, dx \\ \int \cos x \, d\frac{x^2}{2} = \underbrace{\frac{x^2}{2}\cos x - \underbrace{\int \frac{x^2}{2} \, d\cos x}_{\text{worse}}}_{\text{NG}} \end{cases}$$

$$= x\sin x + \cos x + C$$

Check: $\frac{d}{dx}\left(x\sin x + \cos x\right)$

$$= \sin x + x\cos x - \sin x = \cos x$$

(correct)

Eg2 $\int \underbrace{\ln x}_{f} \underbrace{dx}_{dg}$, $(x>0)$

$= x\ln x - \int x \, d\ln x$

$= x\ln x - \int x \frac{1}{x} dx$

$= x\ln x - x + C$

check: $\frac{d}{dx}(x\ln x - x)$

$= x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$

$= \ln x$ (correct)

Eg3. $\int x^2 e^x dx$

$= \int x^2 de^x$

$= x^2 e^x - \int e^x dx^2$

$= x^2 e^x - 2\int x e^x dx$

$= x^2 e^x - 2\int x de^x$

$= x^2 e^x - 2\left(x e^x - \int e^x dx\right)$

$= x^2 e - 2x e^x + 2e^x + C$

Eg4 $\int x^2 \ln x \, dx$, $x>0$

$$= \int \ln x \, d\frac{x^3}{3}$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \, d\ln x$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x} \, dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

<u>Check</u>:

$$\frac{d}{dx} = x^2 \ln x + \frac{x^3}{3}\cdot\frac{1}{x} - \frac{x^2}{3}$$

$$= x^2 \ln x \text{ (correct)}$$

Eg 5 $\int \cos^n x \, dx$, $n \geq 1$

<u>Case 1</u>. $n = 2k+1$.

$$= \int \cos^{2k} x \cos x \, dx$$

$$= \int (1-\sin^2 x)^k \, d\sin x$$

$$= \int (1-s^2)^k \, ds$$

If n = even $\quad c = \cos x$, $s = \sin x$

$$= \int c^{n-1} c \, dx = \int c^{n-1} \, ds$$

$$= c^{n-1} s - \int s \, d(c^{n-1})$$

$$= c^{n-1} s - (n-1) \int s \, c^{n-2} \overbrace{(-s) \, dx}^{dc}$$

$$= C^{n-1}S + (n-1)\int C^{n-2}(1-C^2)dx$$

$$\Rightarrow \int C^n dx = C^{n-1}S + (n-1)\int C^{n-2} dx - (n-1)\int C^n dx$$

$$\Rightarrow n\int C^n dx = C^{n-1}S + (n-1)\int C^{n-2} dx$$

$$\Rightarrow \int \cos^n x\, dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

$$\int C^n dx \to \int C^{n-2} dx \xrightarrow{\cdots} \int C^2 dx \to \int 1\, dx$$

Similarly. (n=even)

$$\int S^n dx \to \int S^{n-2} dx \xrightarrow{\cdots} \int S^2 dx \to \int 1\, dx$$

Eg6: $\int e^x \cos x \, dx$

<u>Sol</u> $= \int e^x \, d\sin x$

$$= e^x \sin x - \int \sin x \underbrace{e^x dx}_{de^x}$$

$$= e^x \sin x + \int e^x \, d(\cos x)$$

$$= e^x \sin x + e^x \cos x - \int \cos x \underbrace{e^x dx}_{de^x}$$

$$\Rightarrow 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + C$$

Definite integrals

$$fg' = (fg)' - gf'$$

$$\int_a^b dx$$

$$\Rightarrow \int_a^b fg'\,dx = fg\Big|_a^b - \int_a^b gf'\,dx$$

or
$$\int_{x=a}^b f\,dg = fg\Big|_a^b - \int_{x=a}^b g\,df$$

Eg 6: $\int_0^4 xe^x\,dx = \int_{x=0}^4 x\,de^x$

$$= xe^x\Big|_0^4 - \int_{x=0}^4 e^x\,dx$$

$$= xe^x\Big|_0^4 - e^x\Big|_0^4 = 3e^4 + 1$$

Trigonometric integrals (8.3)

$$\int \cos^m x \sin^n x \, dx \quad \left(\int C^m S^n dx\right)$$

If $m=2k+1$ $\int C^{2k} S^n C \, dx = \int (1-S^2)^k S^n \, dS$

if $n=2l+1$ $\int C^m S^{2l} S \, dx = -\int C^m (1-C^2)^l \, dC$

If $m=2k$, $n=2l$

Method 1 (k, l small), $C_2 \equiv \cos 2x$

$$\int C^{2k} S^{2l} dx = \int \left(\frac{1+C_2}{2}\right)^k \left(\frac{1-C_2}{2}\right)^l dx$$

$$= \int \left(a_0 + a_1 C_2 + a_2 C_2^2 + \cdots + a_{k+l} C_2^{k+l}\right) dx$$

(reduce from $C^{2k} S^{2l}$ to C_2^{k+l})

Method 2 ($m=2k$, $n=2\ell$)

Case 1: $k \le 1$, $\ell \le 1 \to$ Method 1

Assume $k>1$, we will derive formula $\int C^{2k} S^{2\ell} dx \to \int C^{2k-2} S^{2\ell} dx$

$$= \int C^{2k-1} S^{2\ell} C\, dx = \int C^{2k-1} S^{2\ell}\, dS$$

$$= \frac{1}{2\ell+1} \int C^{2k-1}\, dS^{2\ell+1}$$

$$= \frac{1}{2\ell+1} \left(C^{2k-1} S^{2\ell+1} - \int \underbrace{S^{2\ell+1}\, dC^{2k-1}}_{(*)} \right)$$

$$(*) = (2k-1) \int S^{2\ell+1} C^{2k-2}\, \underset{\substack{\shortparallel \\ -S\,dx}}{dC}$$

$$\therefore \int C^{2k} S^{2\ell}\,dx$$

$$= \frac{1}{2\ell+1}\left(C^{2k-1} S^{2\ell+1} + (2k+1)\left(\int C^{2k-2} S^{2\ell}\,dx - \int C^{2k} S^{2\ell}\,dx \right)\right)$$

$$(S^2 = 1 - C^2)$$

$$\Rightarrow \left(1 + \frac{2k-1}{2\ell+1}\right) \int C^{2k} S^{2\ell}\,dx$$

$$= \frac{1}{2\ell+1} C^{2k-1} S^{2\ell+1} + \frac{2k-1}{2\ell+1} \int C^{2k-2} S^{2\ell}\,dx$$

Reduce from $\int C^{2k} S^{2\ell}\,dx$ to $\int C^{2k-2} S^{2\ell}\,dx$