

8. Techniques of integration

8.2 Integration by parts.

$$\int f(x) g'(x) dx = \int f(x) dg(x) = ?$$

Ans: $\frac{d}{dx} \int f g' dx = f g' = (f g)' - f' g$
 $= \frac{d}{dx} (f g - \int f' g dx)$

$$\Rightarrow \int f g' dx = f g - \int f' g dx$$

$$\text{or } \int f dg = f g - \int g df$$

Useful if $\int g df = \int g f' dx$ is easy to compute.

$$\text{Ex 1 } \int \overset{f'}{x} \overset{f}{\cos x} dx$$

$$= \int x d\sin x = \int x \sin x - \int \sin x dx$$

$$\int \cos x d\frac{x^2}{2} = \underbrace{\frac{x^2}{2} \cos x - \int \frac{x^2}{2} d\cos x}_{\text{NG}} \quad \text{worse}$$

$$= x \sin x + \cos x + C$$

$$\text{Check: } \frac{d}{dx} (x \sin x + \cos x)$$

$$= \sin x + x \cos x - \sin x = \cos x$$

(correct)

$$\text{Eq 2 } \int \underbrace{\ln x}_f \underbrace{dx}_{dg}, (x > 0)$$

$$= x \ln x - \int x d \ln x$$

$$= x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - x + C$$

$$\text{Check: } \frac{d}{dx} (x \ln x - x)$$

$$= x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$$

$$= \ln x \text{ (correct)}$$

$$), \text{ Eg 3. } \int x^2 e^x dx$$

$$= \int x^2 de^x$$

$$= x^2 e^x - \int e^x dx^2$$

$$= x^2 e^x - 2 \int x e^x dx$$

$$= x^2 e^x - 2 \int x de^x$$

$$= x^2 e^x - 2 \left(x e^x - \int e^x dx \right)$$

$$= x^2 e^x - 2 x e^x + 2 e^x + C$$

$$\text{Ex 4 } \int x^2 \ln x \, dx, \quad x > 0$$

$$= \int \ln x \, d\frac{x^3}{3}$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} d \ln x$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

Check:

$$\frac{d}{dx} = x^2 \ln x + \frac{x^3}{3} \cdot \frac{1}{x} - \frac{x^2}{3}$$

$$= x^2 \ln x \text{ (correct)}$$

$$\text{Eg 5. } \int \cos^n x \, dx, n \geq 1$$

Case 1. $n = 2k + 1.$

$$= \int \cos^{2k} x \cos x \, dx$$

$$= \int (1 - \sin^2 x)^k \, d\sin x$$

$$= \int (1 - s^2)^k \, ds$$

If $n = \text{even}$ $\begin{matrix} C = \cos x \\ S = \sin x \end{matrix}$

$$= \int C^{n-1} C \, dx = \int C^{n-1} \, dS$$

$$= C^{n-1} S - \int S \, d(C^{n-1}) \quad dC$$

$$= C^{n-1} S - (n-1) \int S C^{n-2} (-S) \, dx$$

$$= C^{n-1} S + (n-1) \int C^{n-2} (1-C^2) dx$$

$$\Rightarrow \int C^n dx = C^{n-1} S + (n-1) \int C^{n-2} dx - (n-1) \int C^n dx$$

$$\Rightarrow n \int C^n dx = C^{n-1} S + (n-1) \int C^{n-2} dx$$

$$\Rightarrow \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int C^n dx \rightarrow \int C^{n-2} dx \rightarrow \dots \rightarrow \int C^2 dx \rightarrow \int 1 dx$$

Similarly - (n=even)

$$\int S^n dx \rightarrow \int S^{n-2} dx \rightarrow \dots \rightarrow \int S^2 dx \rightarrow \int 1 dx$$

Definite integrals

$$f g' = (f g)' - g f'$$

$$\Rightarrow \int_a^b dx \int_a^b f g' dx = f g \Big|_a^b - \int_a^b g f' dx$$

$$\text{or } \int_{x=a}^b f dg = f g \Big|_a^b - \int_{x=a}^b g df$$

$$\text{Ex 6: } \int_0^4 x e^x dx = \int_{x=0}^4 x d e^x$$

$$= x e^x \Big|_0^4 - \int_{x=0}^4 e^x dx$$

$$= x e^x \Big|_0^4 - e^x \Big|_0^4 = 3e^4 + 1$$

Trigonometric integrals (8.3)

$$\int \cos^m x \sin^n x dx \quad \left(\int C^m S^n dx \right)$$

$$\text{if } m=2k+1 \quad \int C^{2k+1} S^n dx = \int (1-S^2)^k S^n dS$$

$$\text{if } n=2l+1 \quad \int C^m S^{2l+1} dx = \int C^m (1-C^2)^l dC$$

If $m=2k$, $n=2l$.

Method 1 (k, l small), $C_2 = \cos 2x$

$$\int C^{2k} S^{2l} dx = \int \left(\frac{1+C_2}{2} \right)^k \left(\frac{1-C_2}{2} \right)^l dx$$

$$= \int (A_0 + A_1 C_2 + A_2 C_2^2 + \dots + A_{k+l} C_2^{k+l}) dx$$

(reduce from $C^{2k} S^{2l}$ to C_2^{k+l})

Method 2 ($m=2k, n=2l$)

Case 1: $k \leq 1, l \leq 1 \rightarrow$ Method 1

Assume $k > 1$, we will

derive formula $\int C^{2k} S^{2l} dx \rightarrow \int C^{2k-2} S^{2l} dx$

$$= \int C^{2k-1} S^{2l} C dx = \int C^{2k-1} S^{2l} dS$$

$$= \frac{1}{2l+1} \int C^{2k-1} dS^{2l+1}$$

$$= \frac{1}{2l+1} \left(C^{2k-1} S^{2l+1} - \int S^{2l+1} dC^{2k-1} \right)$$

$$(*) = (2k-1) \int S^{2l+1} C^{2k-2} dC$$

= $\int S^{2l+1} C^{2k-2} dx$

$$\therefore \int C^{2k} S^{2l} dx$$

$$= \frac{1}{2l+1} \left(C^{2k-1} S^{2l+1} + (2k+1) \left(\int C^{2k-2} S^{2l} dx - \int C^{2k} S^{2l} dx \right) \right)$$

$(S^2 = 1 - C^2)$

$$\Rightarrow \left(1 + \frac{2k-1}{2l+1} \right) \int C^{2k} S^{2l} dx$$

$$= \frac{1}{2l+1} C^{2k-1} S^{2l+1} + \frac{2k-1}{2l+1} \int C^{2k-2} S^{2l} dx$$

Reduce from $\int C^{2k} S^{2l} dx$ to $\int C^{2k-2} S^{2l} dx$