Hyperbolic functions

(SKIP inverse hyperbolic fcns)

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \left(\begin{matrix}\text{hyperbolic}\\ \text{sine}\end{matrix}\right)$$

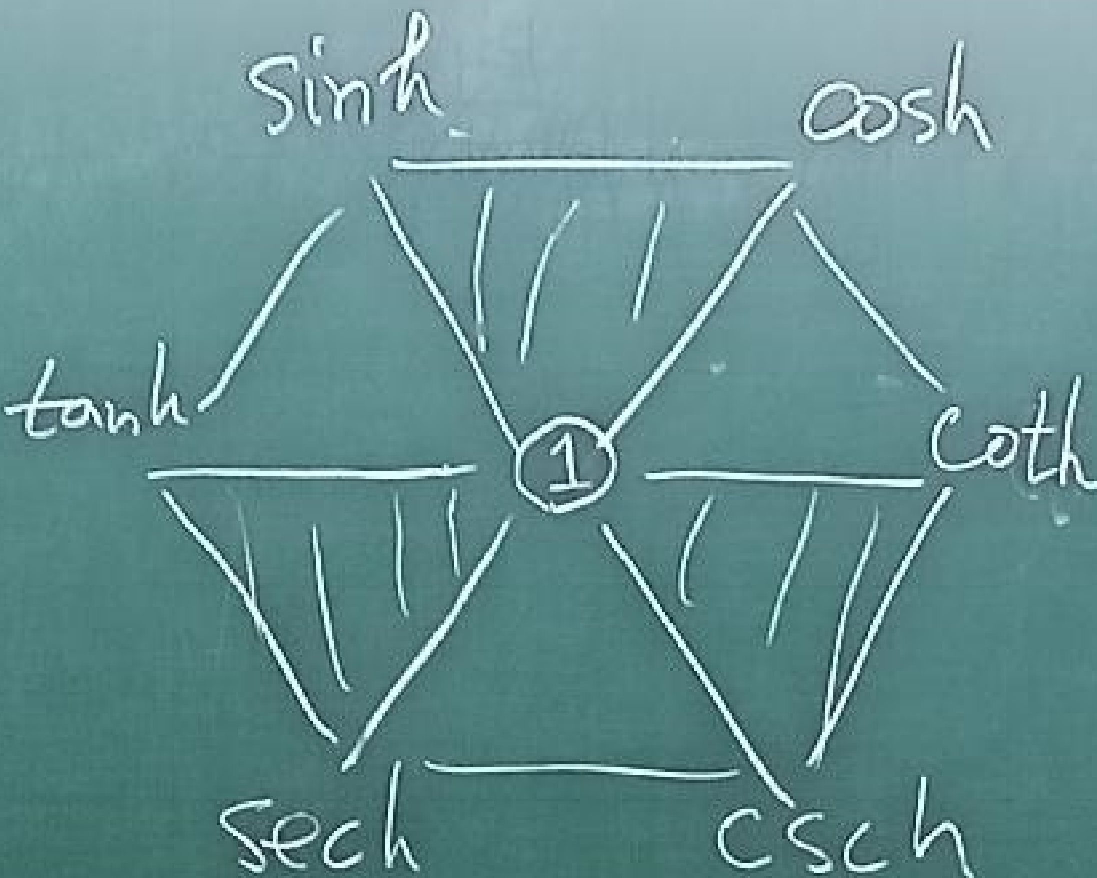
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \left(\begin{matrix}\text{hyp.}\\ \text{cosine}\end{matrix}\right)$$

$$\tanh x = \sinh x / \cosh x$$

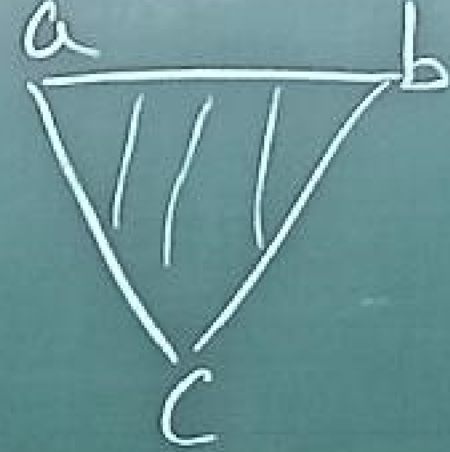
$$\coth x = \cosh x / \sinh x \quad (x \neq 0)$$

$$\operatorname{sech} x = 1/\cosh x$$

$$\operatorname{csch} x = 1/\sinh x \quad (x \neq 0)$$



(i)



$$b^2 - a^2 = c^2$$

(ii) Diagnal entries are reciprocal to each other

(iii) Any corner entry is the product of two neighboring corner entries

Other algebraic identities

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

$$= \frac{(e^{x}+e^{-x})(e^{x}-e^{-x})}{2} = 2\sinh x \cosh x$$

$$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{1}{2}\left((e^{x}+e^{-x})^2 - 2\right) = 2\cosh^2 x - 1$$

$$= \frac{1}{2}\left((e^{x}-e^{-x})^2 + 2\right) = 2\sinh^2 x + 1$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Remark

$$u'' + k^2 u = 0 \Rightarrow u(x) = \begin{matrix} \sin kx \\ \cos kx \end{matrix}$$

$$\begin{cases} u'' + k^2 u = 0 \\ u(0) = 0 \ \underline{(1)} \\ u'(0) = 1 \ \underline{(0)} \end{cases} \Rightarrow u(x) = \frac{1}{k}\sin kx$$

$$(\underline{\cos kx})$$

$$u'' - k^2 u = 0 \Rightarrow u(x) = e^{\pm kx}$$

$$(\text{or } \sinh x, \cosh x)$$

$$\begin{cases} u'' - k^2 u = 0 \\ u(0) = 0 \ \underline{(1)} \\ u'(0) = 1 \ \underline{(0)} \end{cases} \Rightarrow u(x) = \frac{1}{k}\sinh kx$$

$$(\underline{\cosh kx})$$

Derivative of hyperbolic fcns

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}(\cosh x)^{-1} = \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}(\sinh x)^{-1} = \frac{-\cosh x}{\sinh^2 x} = -\coth x \operatorname{csch} x$$

(See also table 7.6)

Eg1 $\frac{d}{dx}\tanh\sqrt{1+x^2}$

$=\left(\operatorname{sech}^2\sqrt{1+x^2}\right)\cdot\frac{x}{\sqrt{1+x^2}}$

Eg2 $\int_0^1 \tanh x\,dx$

$=\int_0^1 \frac{\sinh x}{\cosh x}\,dx$

$=\int_0^1 \frac{d\cosh x}{\cosh x}$

$=\ln|\cosh x|\Big|_0^1$

$=\ln(\cosh 1)-\ln 1=\ln\left(\frac{e+e^{-1}}{2}\right)$

Eg3 $\int \sinh^2 x\,dx$

$\sinh^2 x=\frac{(e^x-e^{-x})^2}{4}$

$=\frac{e^{2x}-2+e^{-2x}}{4}$

$=\frac{1}{2}\left(\left(\frac{e^{2x}+e^{-2x}}{2}\right)-1\right)$

$=\frac{1}{2}(\cosh 2x-1)$.

$\therefore$ Ans $=\frac{1}{2}\int(\cosh 2x-1)\,dx$

$=\frac{1}{2}\left(\frac{1}{2}\sinh 2x-x\right)+C$

$=\frac{1}{4}\sinh 2x-\frac{x}{2}+C$

Eg 4 $\int_0^{\ln 2} 4e^x \sinh x \, dx$

$$= \int_0^{\ln 2} 4 \cdot e^x \cdot \frac{e^x - e^{-x}}{2} \, dx$$

$$= 2\int_0^{\ln 2} e^{2x} - 1 \, dx$$

$$= 2\left(\frac{e^{2x}}{2} - x\right)\Big|_0^{\ln 2}$$

$$= \left(e^{2\ln 2} - 1\right) - 2(\ln 2 - 0)$$

$$= 3 - 2\ln 2$$

Rate of Growth

Def

$f(x)$ grows $\begin{cases} \text{faster than} \\ \text{at the same rate as} \\ \text{slower than} \end{cases}$ $g(x)$

as $x \to \infty$

if $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \begin{cases} \infty \\ L, \quad 0 < L < \infty \\ 0 \end{cases}$

Eg 1 $-5x^5+7x^4-2x^2-1$

grows at the same rate as x^5, as $x \to \infty$

Eg 2: $f(x) = e^{0.01x}$

$g(x) = x^7$, $h(x) = (\ln x)^{1000}$

$f(x)$ grows faster than $g(x)$

$g(x)$ grows faster than $h(x)$

as $x \to \infty$

Since $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$, $\lim_{x\to\infty} \left(\frac{h(x)}{g(x)}\right)^{\frac{1}{1000}} = 0$,

Def. $f(x) = o(g(x))$ as $x \to \begin{matrix} \infty \\ 0^+ \\ 0 \\ \vdots \end{matrix}$

if $\displaystyle\lim_{\substack{x\to\infty \\ 0,\,0^+}} \frac{f(x)}{g(x)} = 0$

(i.e. f "$<$" g)

Def $f(x) = O(g(x))$ as $x \to \begin{matrix} \infty \\ 0^+ \\ 0 \end{matrix} \ldots$

(if $\left|\dfrac{f(x)}{g(x)}\right| \le M$ as $x \to \begin{matrix} \infty \\ 0 \\ 0^+ \end{matrix}$)

if there exists $M>0$, $N>0$, $(\delta>0)$

such that $\left|\dfrac{f(x)}{g(x)}\right| \le M$ for all $\begin{matrix} x>N \\ x\in(-\delta,\delta) \\ x\in(0,\delta) \end{matrix}$

(i.e. f "$\le$" g)

Remark

(i) $\lim \frac{f}{g} = 0 \Leftrightarrow f = o(g)$

(ii) $\lim \left|\frac{f}{g}\right| = \underset{\neq 0}{L} \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

(iii) $\lim \left|\frac{f}{g}\right| = \begin{cases} 0 \\ L \end{cases} \Longrightarrow f = O(g)$

((iv) $f = o(g) \Rightarrow f = O(g)$)

Eg. $g(x) = x$, $f(x) = x(2+\sin x)$

$\Rightarrow \left|\frac{f(x)}{g(x)}\right| = 2 + \sin x \begin{array}{l}\leq 3 \\ \geq 1\end{array} \Rightarrow \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

But $\lim\limits_{x \to \infty} \left|\frac{f(x)}{g(x)}\right|$ does not exist as $x \to \infty$

Eg 3

$$-5x^5+7x^4-2x-1=O(x^5)$$

as $x\to\infty$

Eg 4 $$(\ln x)^{1000}=o(x^7)$$

$$x^7=o(e^{0.001x}) \quad \text{as } x\to\infty$$

Eg 5 $$(\ln x)^{1000}=o(x^{-2})$$

$$x^{-2}=o\left(e^{\frac{1}{x^2}}\right) \quad \text{as } x\to 0^+$$

Eg 6 $x^p = o(x^q)$

(i) as $x \to \infty$ if $p < q$

(ii) as $x \to 0$ if $p > q$

Eg 7. $x + \sin x = O(x)$ as $x \to \infty$

Eg 8: $e^x + x^3 = O(e^x)$

as $x \to \infty$