First order linear differential equations

(textbook: $p, q \longrightarrow P, Q$)

$$\frac{dy}{dx} = \underbrace{-p(x)y + q(x)}_{\text{linear in } y}$$

(Solve for $y(x)$)

Eg1: $y' = 3y$, $y(0) = 2$

Sol: $e^{-3x} \cdot (y' - 3y = 0)$

$\Rightarrow (e^{-3x}y)' = 0$, $e^{-3x}y(x) = C$

$y(0) = 2 \Rightarrow C = 2$, $\therefore y(x) = 2e^{3x}$

In general

$$\frac{dy}{dx}+p(x)y=q(x),\ y(a)=y_0$$

Multiply by $e^{\mathbb{P}(x)}$, where $\mathbb{P}'(x)=p(x)$

$$\left(e^{\mathbb{P}(x)}y\right)'=e^{\mathbb{P}(x)}q(x)$$

$$\therefore e^{\mathbb{P}(x)}y(x)=\int e^{\mathbb{P}(x)}q(x)\,dx=\int^x e^{\mathbb{P}(t)}q(t)\,dt$$

$y(a)=y_0 \Rightarrow$ find the undetermined constant in $\int e^{\mathbb{P}(x)}q(x)\,dx$

or

$$\int_a^x\left(e^{\mathbb{P}(t)}y(t)\right)'dt=\int_a^x e^{\mathbb{P}(t)}q(t)\,dt$$

$$\therefore e^{\mathbb{P}(x)}y(x)-e^{\mathbb{P}(a)}y(a)=\int_a^x e^{\mathbb{P}(t)}q(t)\,dt$$

$$\therefore y(x)=e^{-(\mathbb{P}(x)-\mathbb{P}(a))}y_0+e^{-\mathbb{P}(x)}\int_a^x e^{\mathbb{P}(t)}q(t)\,dt$$

Eg2 Solve $e^{x}y' + 2e^{2x}y = 1$

Sol: $\frac{dy}{dx} + 2e^{x}y = e^{-x}$

Multiply by $e^{2e^{x}}$

$\Rightarrow \left(e^{2e^{x}}y\right)' = e^{2e^{x}-x}$

$\therefore e^{2e^{x}}y(x) = \int e^{2e^{x}-x}\,dx$

$y(x) = e^{-2e^{x}}\int e^{2e^{x}-x}\,dx$

$= e^{-2e^{x}}\int^{x} e^{2e^{t}-t}\,dt$

Eg3. Solve $\begin{cases} y' + xy = x \\ y(0) = 2 \end{cases}$

Sol Multiply by $e^{\frac{x^2}{2}}$

$$\left(e^{\frac{x^2}{2}} y\right)' = x e^{\frac{x^2}{2}}$$

$\int_0^x \cdots dt$

$$\int_0^x \left(e^{\frac{t^2}{2}} y(t)\right)' dt = \int_0^x t e^{\frac{t^2}{2}} dt$$

$$e^{\frac{x^2}{2}} y(x) - y(0) = \int_0^x e^{\frac{t^2}{2}} d\left(\frac{t^2}{2}\right)$$

$$e^{\frac{x^2}{2}} y(x) = y(0) + e^{\frac{x^2}{2}} - e^0$$

$$y(x) = e^{\frac{-x^2}{2}} + 1$$

Eg4 $\begin{cases} (1+x)y' + y = \sqrt{x} \\ y(0) = 1 \qquad (x \geq 0) \end{cases}$

Sol $y' + \frac{1}{1+x} y = \frac{\sqrt{x}}{1+x}$

$p(x) = \frac{1}{1+x}$. $\mathbb{P}(x) = \ln(1+x)$

$e^{\mathbb{P}(x)} = (1+x)$

$((1+x)y)' = \sqrt{x}$

$\int_0^x [(1+t)y(t)]' dt = \int_0^x \sqrt{t}\, dt$

$(1+x)y(x) - (1+0)y(0) = \frac{2}{3}x^{\frac{3}{2}}$

$\therefore y(x) = \dfrac{\frac{2}{3}x^{\frac{3}{2}} + 1}{1+x}$

Eg 5 $\begin{cases} x\dfrac{dy}{dx} = x^2 + 3y, & x > 0 \\ y(1) = 1 \end{cases}$

Sol $xy' - 3y = x^2$

$$y' - \frac{3}{x}y = x$$

$$(x^{-3}y)' = -3x^{-4}y + x^{-3}y'$$

$$= x^{-4}(xy' - 3y)$$

$$xy' - 3y = x^4(x^{-3}y)'$$

$$\Rightarrow (x^{-3}y)' = x^{-2}$$

$$\int_1^x dt \Rightarrow x^{-3}y(x) - 1 \cdot y(1) = \int_1^x t^{-2}\,dt$$

$$\Rightarrow y(x) = -x^2 + 2x^3$$

In general, the special case.
(k is an integer)

$$xy' + ky = g(x)$$
$$\parallel$$
$$x^{(1-k)}(x^k y)'$$

$$\Rightarrow (x^k y)' = x^{k-1} g(x)$$

i.e. In this case

$$p(x) = \frac{k}{x}, \Rightarrow \mathbb{P}(x) = k \ln|x|$$

Integration factor

$$= e^{\mathbb{P}(x)} = e^{k \ln|x|} = |x|^k = \pm x^k$$

i.e. $x^k \cdot \left(y' + \frac{k}{x} y = \frac{g(x)}{x}\right)$

Eg6. $y' + 2y = 3$, $y(0) = 1$

Sol. $e^{2x}(y' + 2y) = 3e^{2x}$

$$\Rightarrow (e^{2x}y)' = 3e^{2x}$$

$\int_0^x dt$

$$\Rightarrow e^{2x}y(x) - e^{0}y(0) = 3\int_0^x e^{2t}\,dt$$

$$\therefore y(x) = e^{-2x}\left(1 + 3 \cdot \frac{e^{2t}}{2}\Big|_0^x\right)$$

$$= \frac{-1}{2}e^{-2x} + \frac{3}{2}$$