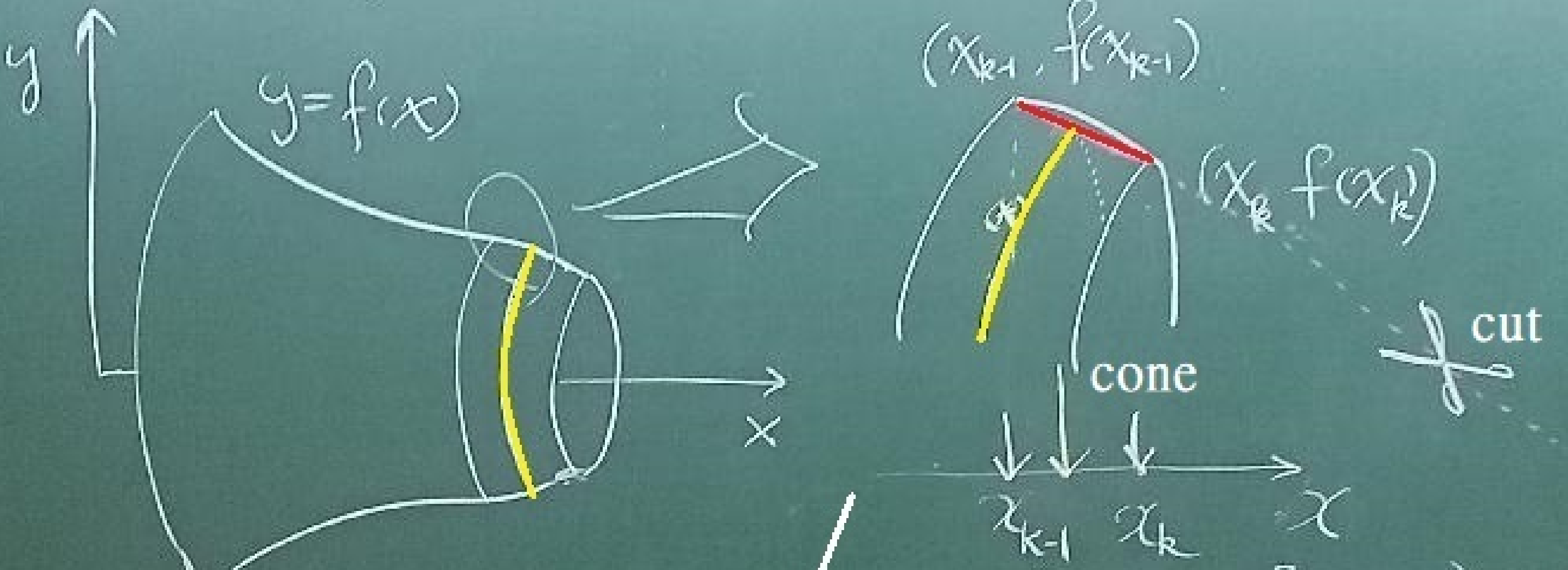
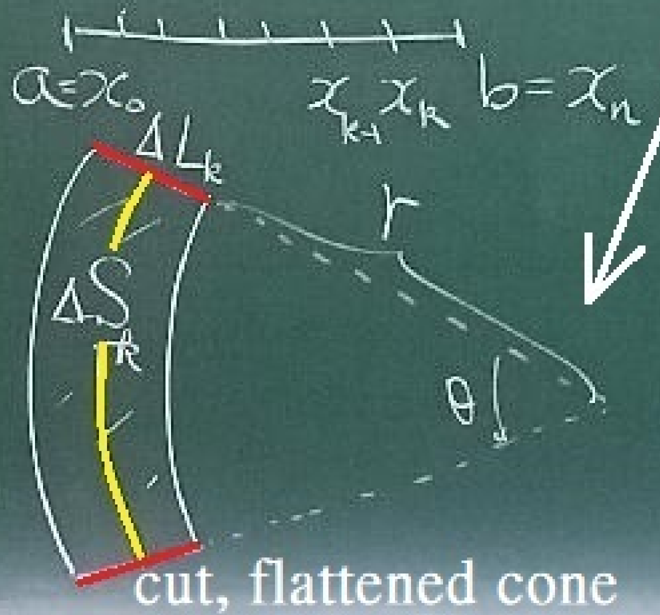
Area of Surface of Revolution



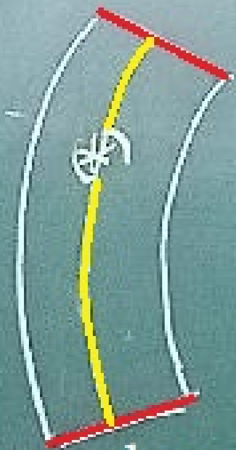


$$\Delta S_k = \left(\pi(r+\Delta L_k)^2 - \pi r^2\right)\cdot\frac{\theta}{2\pi}$$

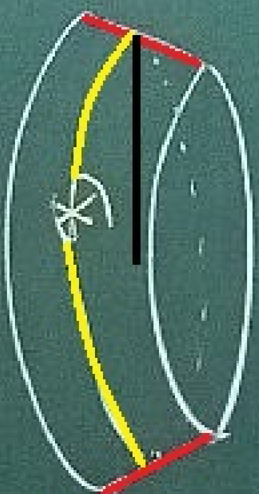
$$= \pi\left(2\cancel{\pi} r\,\Delta L_k + \Delta L_k^2\right)\cdot\frac{\theta}{2\pi}$$

$$= \underline{\Delta L_k}\,\underline{\left(2\pi\left(r+\frac{\Delta L_k}{2}\right)\right)\frac{\theta}{2\pi}}$$

$$= \Delta L_k \cdot \tilde{L}_k$$



flattened cone



cone

$$\Delta L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$\tilde{L}_k$ = arclength of (*)

$$= 2\pi f\Big(\underbrace{\frac{x_{k-1}+x_k}{2}}_{\equiv c_k}\Big)$$

$$S = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta S_k$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi f(c_k)\,\Delta L_k$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi f(c_k)\sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

$$= \int_a^b 2\pi \underbrace{f(x)}_{\text{半徑}} \underbrace{\sqrt{1+f'(x)^2}\,dx}_{\text{弧長元}}$$

Remark For a surface obtained by rotating $\left\{ \begin{array}{l} c \le y \le d \\ x = g(y) \end{array} \right\}$ around y-axis, the surface

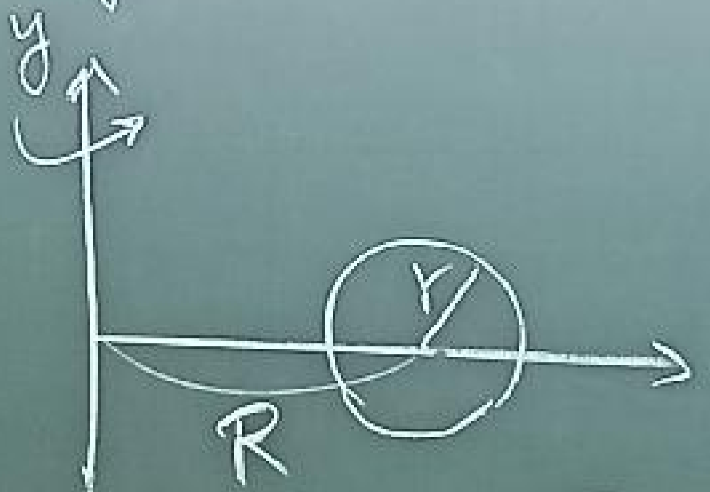
$$\text{area} = \int_c^d 2\pi g(y) \sqrt{1 + g'(y)^2}\, dy$$

Rm $\left\{ \alpha \le t \le \beta,\ \begin{array}{l} x = X(t) \\ y = Y(t) \end{array} \right\}$

$$S_{\rightarrow} = \int_\alpha^\beta 2\pi Y(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

$$S_{\uparrow} = \int_\alpha^\beta 2\pi X(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

Eg1 Surface area of a donut



generated by rotating $\{(x-R)^2+y^2=r^2\}$ around y-axis.

半徑 弧長元

$$S^{上+下}=2\int_{x=R-r}^{R+r} 2\pi x \, d\ell$$

$\left(y=\pm\sqrt{r^2-(x-R)^2}\right)$ $\quad d\ell=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$\frac{dy}{dx}=\pm\left(-\frac{(x-R)}{\sqrt{r^2-(x-R)^2}}\right)$$

$$\therefore d\ell=\sqrt{1+\frac{(x-R)^2}{r^2-(x-R)^2}}\,dx=\frac{r}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$\therefore S^{\text{上}+\text{下}} = 2\int_{R-r}^{R+r} 2\pi x \frac{r}{\sqrt{r^2-(x-R)^2}}\,dx$$

Let $\mathbb{X} = x - R \quad dx = d\mathbb{X}$

$$= 2\int_{\mathbb{X}=-r}^{r} 2\pi(\mathbb{X}+R)\frac{r}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}$$

$$= 4\pi r\left(\int_{-r}^{r}\frac{\mathbb{X}}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X} + \int_{-r}^{r}\frac{R}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}\right)$$

$$= 4\pi r\left(\frac{-1}{2}\int_{\mathbb{X}=-r}^{r}\frac{d(r^2-\mathbb{X}^2)}{\sqrt{r^2-\mathbb{X}^2}} + R\int_{\mathbb{X}=-r}^{r}\frac{1}{\sqrt{1-\left(\frac{\mathbb{X}}{r}\right)^2}}\,d\left(\frac{\mathbb{X}}{r}\right)\right)$$

$$= 4\pi r\left(-\sqrt{r^2-\mathbb{X}^2}\,\Big|_{\mathbb{X}=-r}^{r} + R\sin^{-1}\left(\frac{\mathbb{X}}{r}\right)\Big|_{\mathbb{X}=-r}^{r}\right)$$

$$= 4\pi^2 rR = (2\pi r)\cdot(2\pi R)$$

Separable differential equations

First order differential equations

$$\boxed{\text{Solve for } y(x) \text{ from } \quad \frac{dy(x)}{dx} = F(x, y(x))}$$

Eg: $\begin{cases} y'(x) = \sin(x+y) \\ y(0) = y_0 \end{cases}$

Special cases

(I): Seperable: $F(x,y) = g(x) H(y)$

(II): linear: $F(x,y) = -P(x) y + Q(x)$

(linear in y)

Eg 1 (Seperable) Solve y(x) from

$$\frac{dy}{dx} = (1+y)e^x \quad (y > -1)$$

Ans:

$$\frac{\frac{dy}{dx}}{1+y} = e^x \qquad \frac{dy}{1+y} = e^x dx$$

$$\int dx \Rightarrow \int \frac{\frac{dy}{dx}}{1+y} dx = \int e^x dx$$

i.e. $$\int \frac{dy}{1+y} = \int e^x dx$$

$$\ln(1+y) \overset{y>-1}{=} \ln|1+y| = e^x + C$$

$$\Rightarrow 1+y = e^{(e^x+C)} = C_1 e^{e^x} \quad (C_1 = e^C)$$

$$\therefore y = C_1 e^{e^x} - 1 \quad \left(\begin{array}{c}C_1 \text{ to be determined} \\ \text{from additional BC}\end{array}\right)$$

Eg2: Solve $y \cdot (x+1) \frac{dy}{dx} = x(y^2+1)$

$$\left(\frac{y}{y^2+1} dy = \frac{x}{x+1} dx \right)$$

Sol $\int \frac{y}{1+y^2} dy = \int \frac{x}{x+1} dx$

$$\Rightarrow \frac{1}{2} \int \frac{dy^2}{1+y^2} = \int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \frac{1}{2} \int \frac{d(1+y^2)}{1+y^2} = x - \int \frac{dx}{x+1}$$

$$\Rightarrow \frac{1}{2} \ln(1+y^2) = x - \ln|1+x| + C$$

$$\Rightarrow \ln(1+y^2) = 2x - \ln(1+x)^2 + 2C$$

$$y = \pm \sqrt{\frac{e^{2(x+C)}}{(1+x)^2} - 1} \quad \left(\begin{array}{c} \text{2 sets of} \\ \text{solutions} \end{array} \right)$$