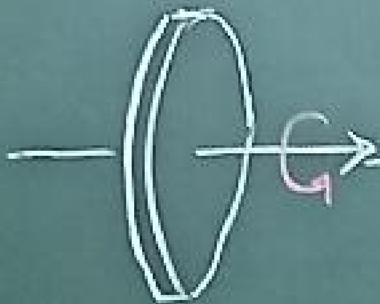
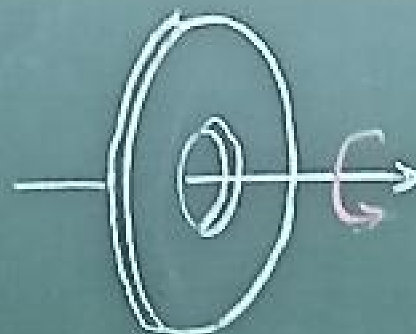
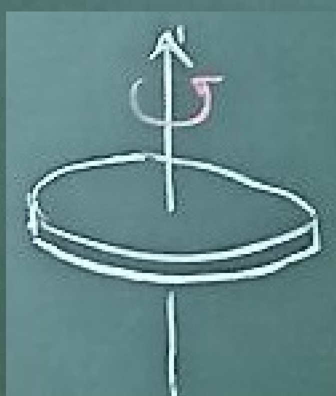
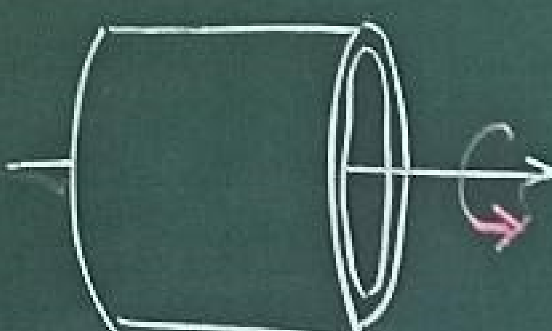
Disks (Washers)

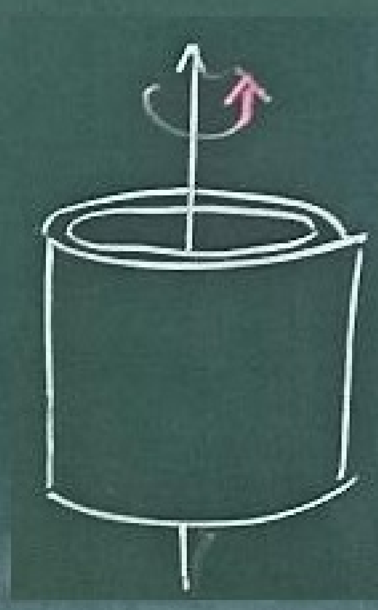






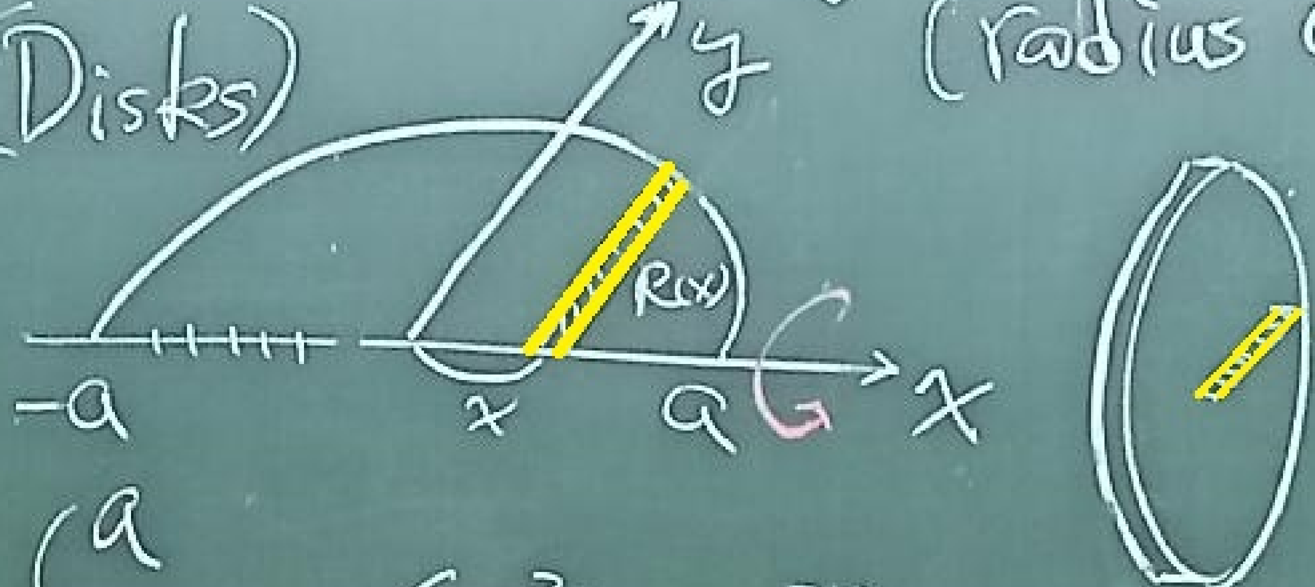
Cylindrical Shells





Eg] Volume of a ball

(Disks) (radius a)



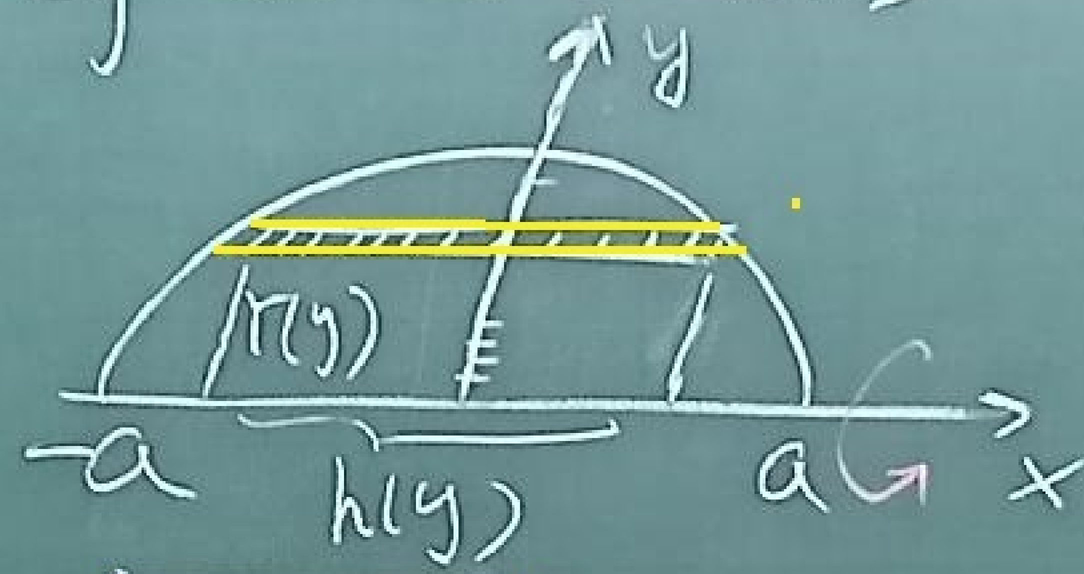
$$V=\int_{-a}^{a}\pi\left(R^2(x)-0^2\right)dx$$

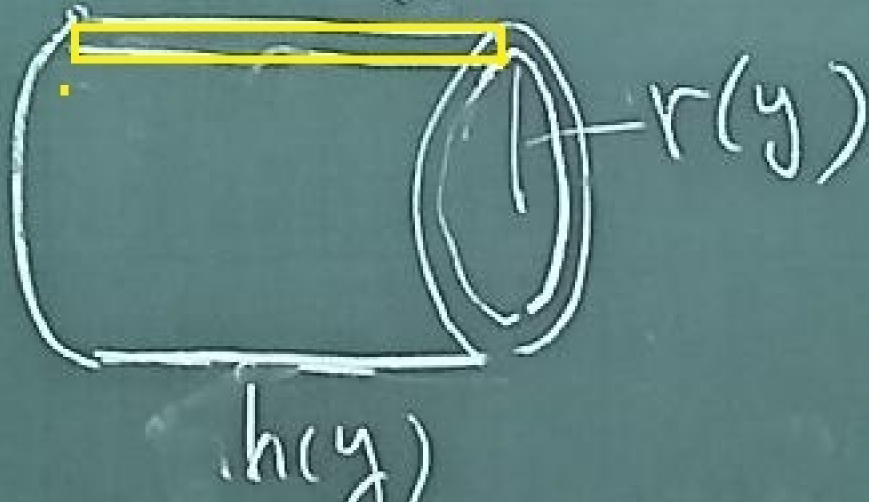
$$x^2+R^2(x)=a^2$$

$$=\int_{-a}^{a}\pi\left(a^2-x^2\right)dx$$

$$=\frac{4}{3}\pi a^3$$

Cylindrical Shells





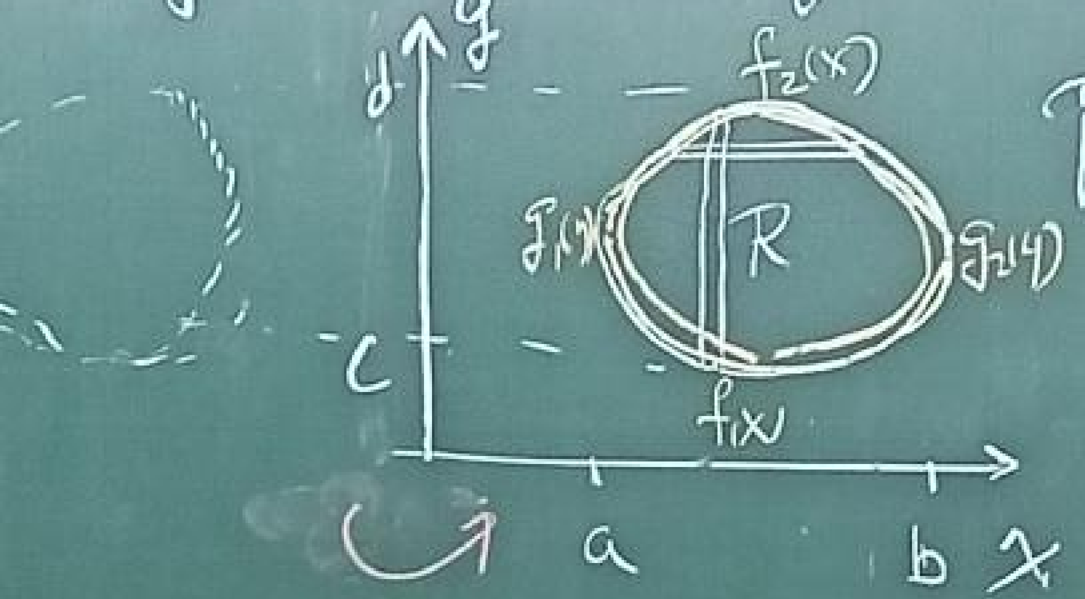
$$V = \int_{y=0}^{a} 2\pi r(y)\, h(y)\, dy$$

$$r(y) = y, \quad h(y) = 2\sqrt{a^2 - y^2}$$

$$= \pi \int_0^a 4y\sqrt{a^2-y^2}\, dy$$

$$= -2\pi \int_0^a \sqrt{a^2-y^2}\, d(a^2-y^2)$$

Eg2. Volume of a donut

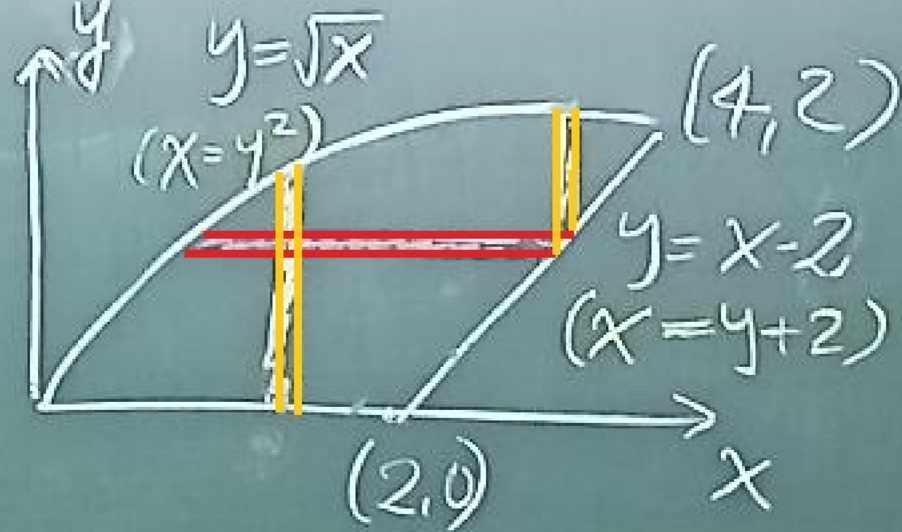


$$R=\left\{\begin{array}{l}0<a\le x\le b\\ f_1(x)\le y\le f_2(x)\end{array}\right\}=\left\{\begin{array}{l}c\le y\le d\\ 0<g_1(y)\le x\le g_2(y)\end{array}\right\}$$

$$V \overset{\text{disks}}{=} \int_{y=c}^{d} \pi\left(\underset{外圓}{g_2^2(y)}-\underset{內圓}{g_1^2(y)}\right)\underset{厚}{dy}$$

$$\overset{\text{Shells}}{=} \int_a^b \underbrace{2\pi \underset{半徑}{x}}_{圓周長}\underbrace{\left(f_2(x)-f_1(x)\right)}_{高度}\underset{厚}{dx}$$

Eg 3



$$V \overset{\text{Shells}}{=\!=} \int_0^2 2\pi y\left(y+2-y^2\right)dy$$

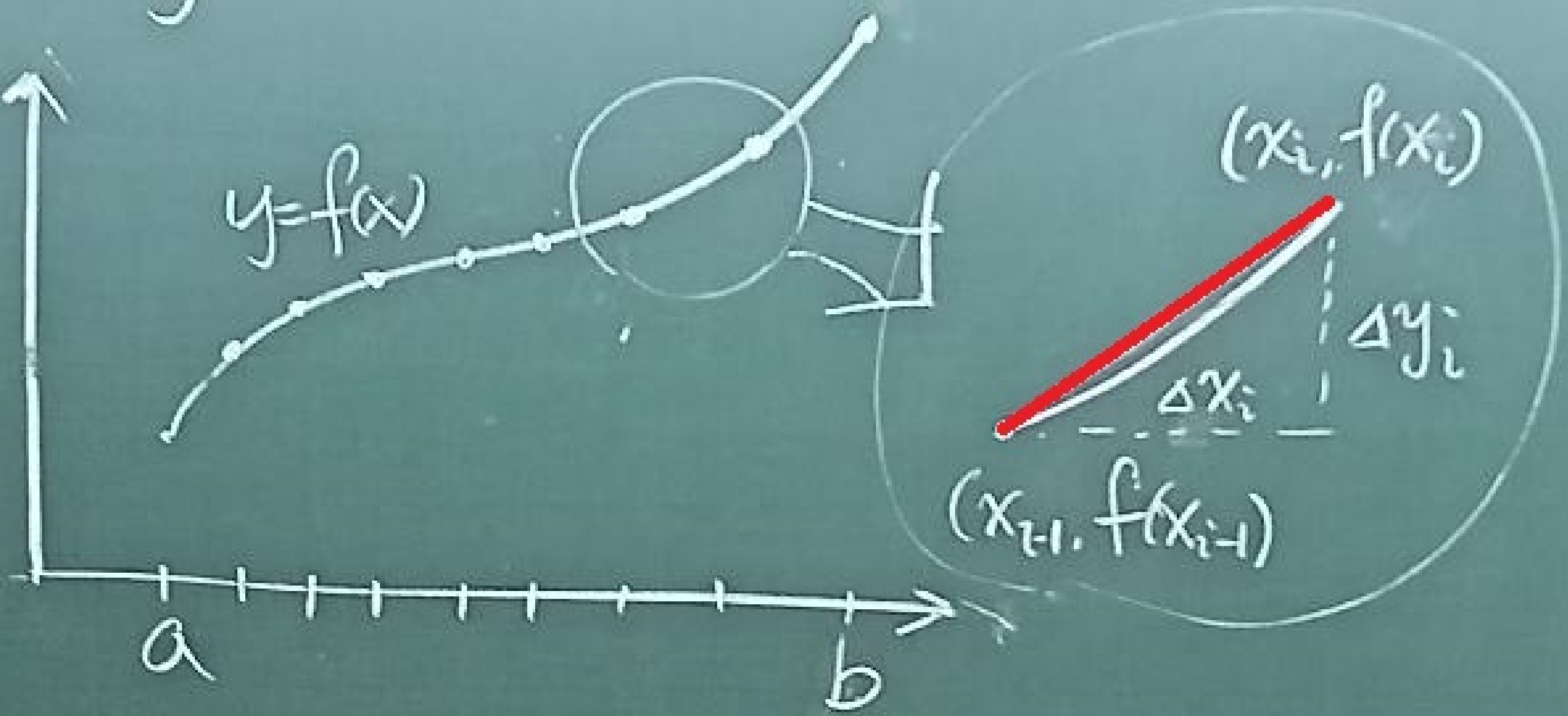
$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\sqrt{x}^2\,dx + \int_2^4 \pi\left(\sqrt{x}^2-(x-2)^2\right)dx$$

$$V \overset{\text{Shell}}{=\!=} \int_0^2 2\pi x\left(\sqrt{x}-0\right)dx$$

$$+\int_2^4 2\pi x\left(\sqrt{x}-(x-2)\right)dx$$

$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\left((y+2)^2-(y^2)^2\right)dy$$

Arclength



$$L_i \cong \underline{\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}} = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_i = f(x_i) - f(x_{i-1})$$

$$= \int_{x=a}^{b} \sqrt{1+(f'(x))^2}\,dx$$

Rm If the curve is

$$\{c \le y \le d,\ x = g(y)\}$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta y_i$$

$$= \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Rm If the curve is

$$\left\{\alpha \le t \le \beta,\ \begin{array}{l} x = X(t) \\ y = Y(t) \end{array}\right\}$$

$$L_i \cong \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i = \int_\alpha^\beta \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

Eg1 Perimeter of $\{x^2+y^2=a^2\}$

Method 1 $y=\pm\sqrt{a^2-x^2}$

$$L=2\int_{-a}^{a}\sqrt{1+\left(\frac{d}{dx}\sqrt{a^2-x^2}\right)^2}\,dx$$

$$=2\int_{-a}^{a}\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx$$

$$=2a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=2a\int_{x=-a}^{a}\frac{d\frac{x}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$=2\pi a$$

Method 2: $L=2\int_{y=-a}^{a}\sqrt{1+\frac{y^2}{a^2-y^2}}\,dy$

Method 3 $x=a\cos t,\ y=a\sin t,\ 0\le t\le 2\pi$

$x'=-a\sin t,\ y'=a\cos t$

$$L=\int_0^{2\pi}\sqrt{(-a\sin t)^2+(a\cos t)^2}\,dt=2\pi a$$