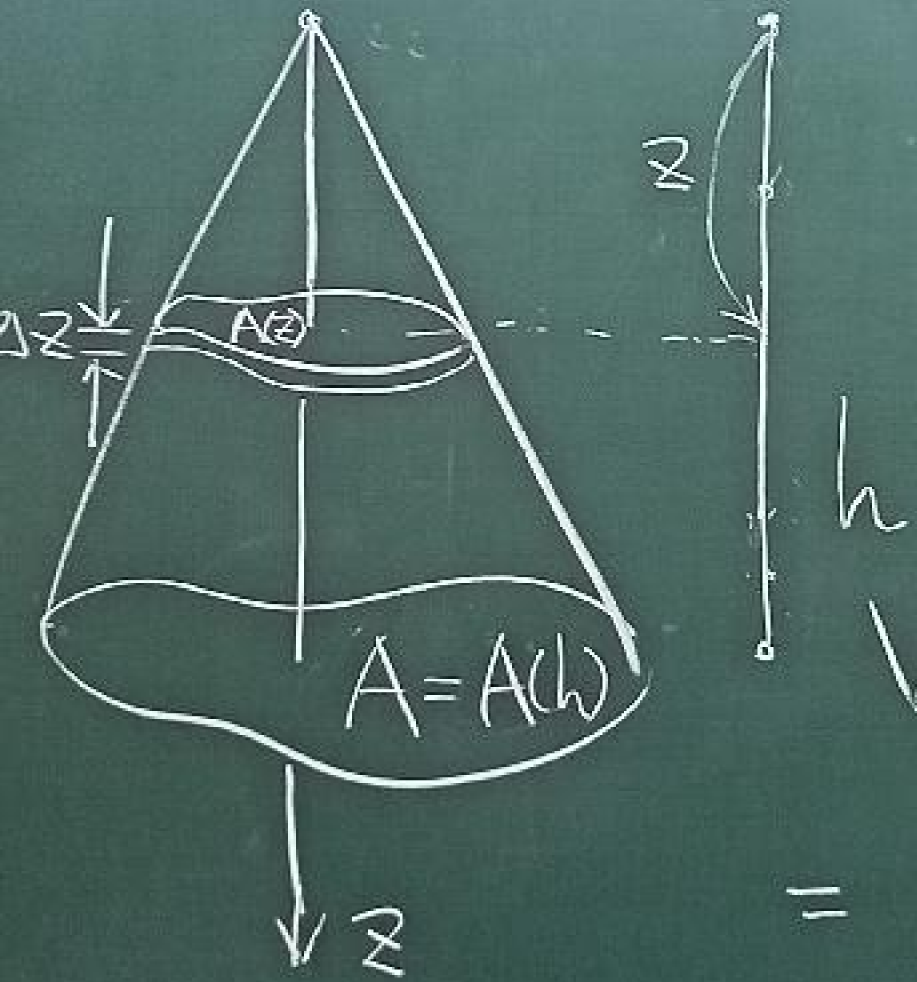
Volume by Cross Section

Eg 1



$$\frac{A(z)}{A(h)} = \left(\frac{z}{h}\right)^2$$

$$A(z) = \frac{A}{h^2} z^2$$

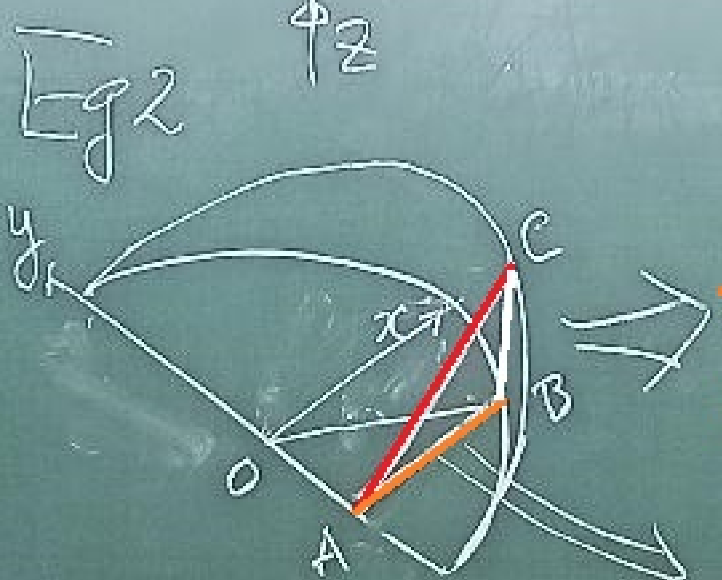
$$\Delta V(z) \cong A(z)\,\Delta z$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(z_k)\,\Delta z_k$$

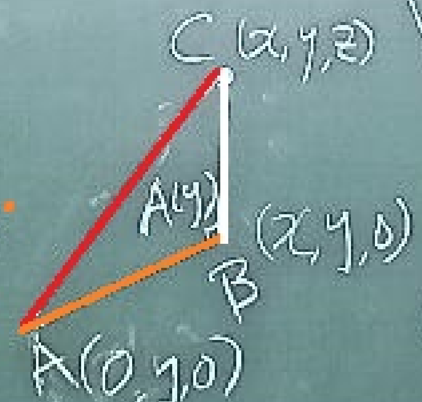
$$= \int_0^h A(z)\,dz = \int_0^h \frac{A}{h^2} z^2\,dz$$

$$= \frac{1}{3} A h$$

Eg2



Method I: cut along "y = constant"



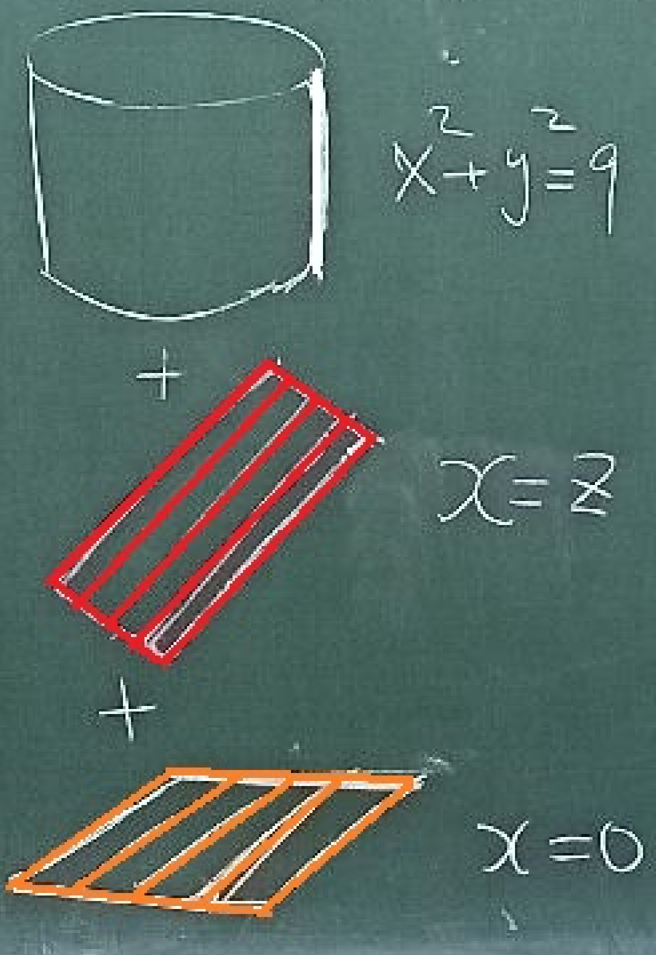
$$V = \int_{-3}^{3} A(y)\,dy$$

$$A(y) = \frac{1}{2}\,\overline{AB}\cdot\overline{BC}$$

$$\Rightarrow \overline{AB} = \sqrt{3^2 - y^2}$$

$$\overline{BC} = z = x = \overline{AB}$$

$$\therefore \overline{BC} = \overline{AB} = \sqrt{9 - y^2}$$

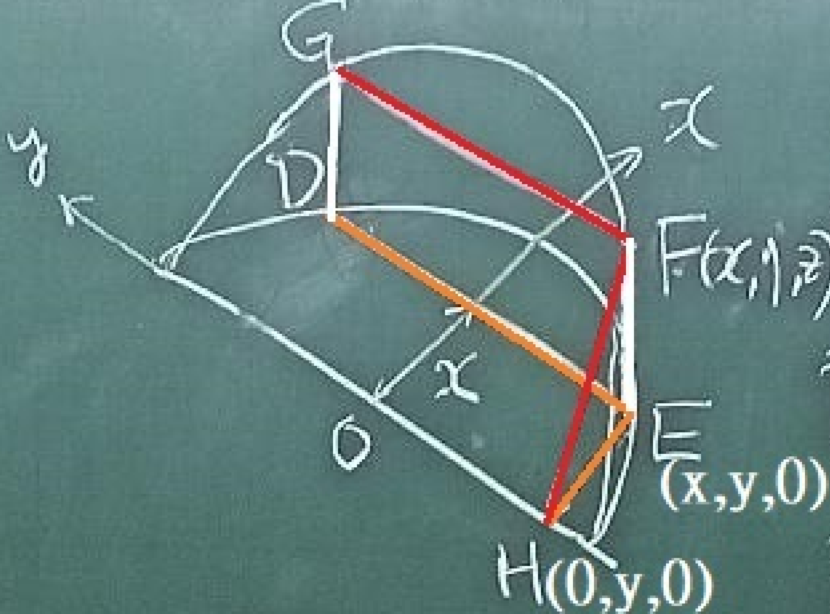


$$A(y) = \frac{1}{2}(9 - y^2)$$

$$V = \int_{-3}^{3} \frac{1}{2}(9 - y^2) = \left.\frac{9y}{2} - \frac{y^3}{6}\right|_{-3}^{3}$$

$$= 18$$

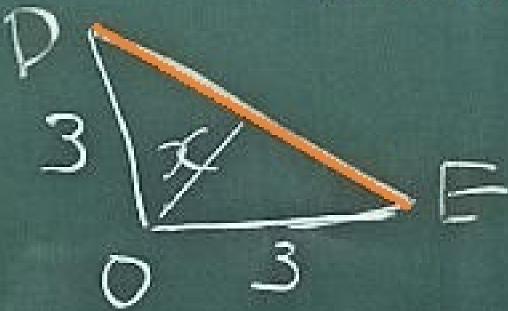
Method II, Cut along "$x = \text{const}$"



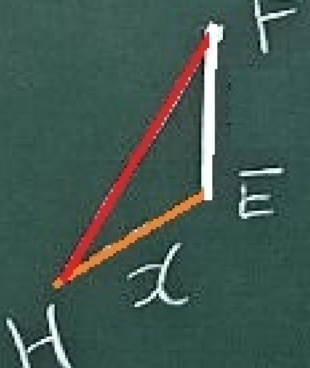
$$A(x) = \overline{DE} \cdot \overline{EF}$$

$$\overline{DE} = 2\sqrt{3^2 - x^2}$$

$$\overline{EF} = \overline{HE} = x$$



$$V = \int_0^3 A(x)\,dx$$



$$= \int_0^3 2x\sqrt{9-x^2}\,dx$$

$$= \int_0^3 \sqrt{9-x^2}\,dx^2$$

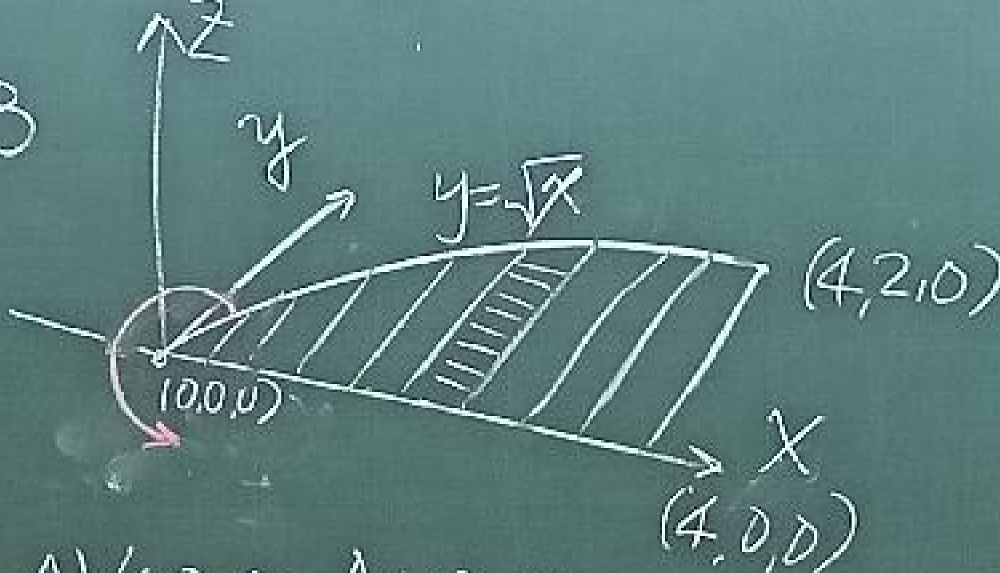
$$= -\int_0^3 \sqrt{9-x^2}\,d(9-x^2)$$

$$= -\frac{2}{3}(9-x^2)^{\frac{3}{2}}\Big|_0^3 = 18$$

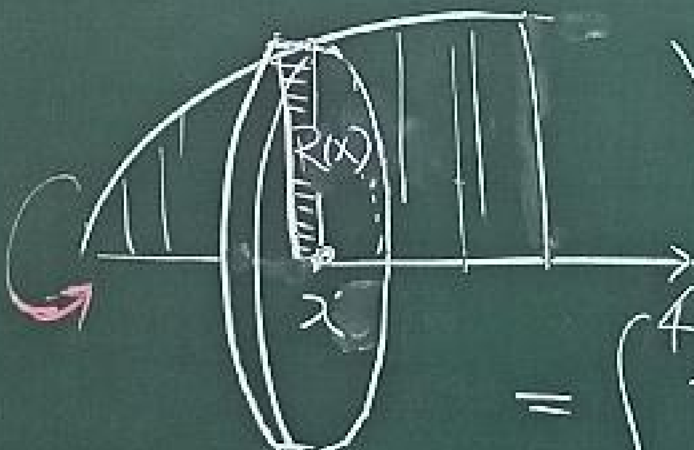
Volume of Revolution

(I). Method of disks

Eg 3



$$\Delta V(x) \cong A(x)\Delta x$$



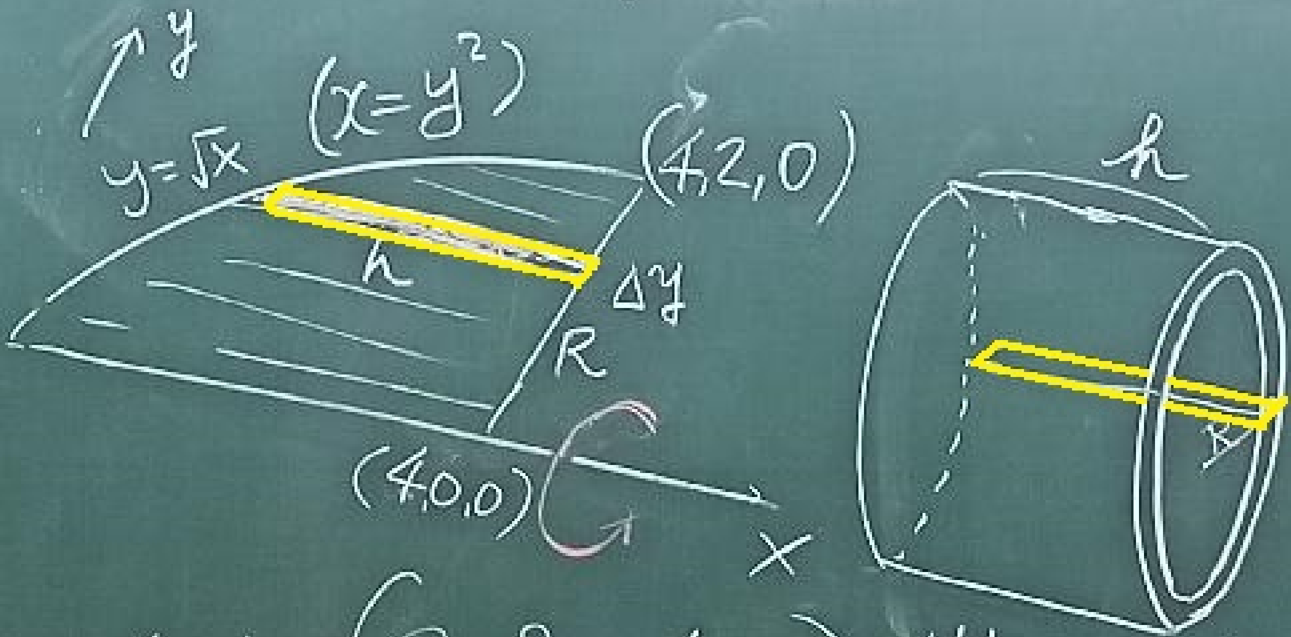
$$V = \int_0^4 A(x)\,dx$$

$$= \int_0^4 \pi R(x)^2\,dx$$

Here $R(x) = y = \sqrt{x}$

$$\therefore V = \int_0^4 \pi x\,dx = 8\pi$$

(II) Method of Cylindrical Shells



$$\Delta V = (\text{Surface Area}) \cdot (\text{thickness})$$

$$= 2\pi R(y)\, h(y) \cdot \Delta y$$

$$= 2\pi y (4 - y^2)\, \Delta y$$

$$V = \int_0^2 2\pi y (4 - y^2)\, dy$$

$$= 2\pi \left(2y^2 - \frac{y^4}{4} \right) \Big|_0^2$$

$$= 8\pi$$