The Substitution Rule

If g is differentiable and f is cont. on the range of g, then

$$\int f(g(x))\,g'(x)\,dx$$

$$= \int \frac{d}{dx} F(g(x))\,dx \,, \quad \left(F'(u) = f(u) \right)$$

$$= F(g(x)) + C$$

$$\int_{x=a}^{b} f(g(x))\,g'(x)\,dx = F(g(x)) \Big|_{x=a}^{b}$$

Eg 1 $\int_0^1 \underbrace{(x^3+x)}_{g(x)}{}^5 \underbrace{(3x^2+1)}_{g'(x)} dx$

$$= \frac{1}{6}\int_0^1 \frac{d}{dx}(x^3+x)^6 \, dx$$

$$= \frac{1}{6}(x^3+x)^6 \Big|_0^1$$

$$= \frac{64}{6} - \frac{0}{6} = \frac{32}{3}$$

Rm $g' dx = dg$ $\left(\Leftarrow g' = \frac{dg}{dx}\right)$

Since $\int F'(g(x))\, g'(x)\, dx$

$= F(g(x)) + C = \int F'(g)\, dg$

Eg2 $\displaystyle\int_0^1 \underbrace{\sqrt{2x+1}}_{g^{\frac{1}{2}}}\,dx$

$$= \frac{1}{2}\int_0^1 \sqrt{2x+1}\cdot \overbrace{2}^{g'}\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1} g^{\frac{1}{2}}(x)\,\overbrace{g'(x)\,dx}^{dg} \qquad \left(\begin{array}{l} g' = \frac{dg}{dx} \\ g'dx = dg \end{array}\right)$$

$$g(x) = 2x+1$$

$$\left(= \frac{1}{2}\int_{g=1}^{3} g^{\frac{1}{2}}\,dg = \frac{1}{3}\,g^{\frac{3}{2}}\Big|_{g=1}^{3} \right)$$

$$= \int_{x=0}^{1} \frac{1}{3}\frac{d}{dx}(2x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_{x=0}^{1} = \frac{1}{3}\left(3\sqrt{3}-1\right)$$

Other examples:

$$\int f(\sin x)\cos x\,dx = \int f(\sin x)\,d\sin x = F(\sin x) + C$$

$$\int f(\cos x)\sin x\,dx = \int f(\cos x)\,d(-\cos x) = -F(\cos x) + C$$

$$\int e^{g(x)} g'(x)\,dx = \int e^{g(x)}\,dg(x) = e^{g(x)} + C$$

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{1}{g(x)}\,dg(x) = \ln|g(x)| + C$$

Eg3 $\int \sec^2(7x+3)\,dx = \frac{\tan(7x+3)}{7} + C$

Eg4 $\int_0^1 x^2 e^{x^3}\,dx = \frac{1}{3} e^{x^3}\Big|_{x=0}^{1} = \frac{e-1}{3}$

Eg5 $\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^{2x}+1} e^x\,dx$

$= \int \frac{1}{1+e^{-2x}} e^{-x}\,dx$

$= \int \frac{1}{1+(e^{-x})^2}\,d(-e^{-x})$

$= \underbrace{-\tan^{-1}(e^{-x})+C}_{(\mathrm{II})}$

$= \int \frac{1}{1+(e^x)^2}\,de^x$

$= \underbrace{\tan^{-1}(e^x)+C}_{(\mathrm{I})}$

Rm $\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2}$ $(a>0,\ b>0)$

$\Rightarrow (\mathrm{I}) = (\mathrm{II})$

Eg 6 $\int \frac{\cos x}{1+\sin^2 x}\,dx$

$$= \int \frac{d\sin x}{1+\sin^2 x}$$

$$= \tan^{-1}(\sin x) + C.$$

Eg 7 $\int \sec x\,dx$

$$= \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx$$

$$= \int \frac{d\sin x}{1-\sin^2 x} = \int \frac{ds}{1-s^2}$$

$$= \frac{1}{2}\int \left(\frac{1}{1+s} + \frac{1}{1-s}\right) ds$$

$$= \frac{1}{2}\int \frac{d(1+s)}{1+s} - \frac{1}{2}\int \frac{d(1-s)}{1-s}$$

$$= \frac{1}{2}\ln|1+s| - \frac{1}{2}\ln|1-s| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+s}{1-s}\right| + C$$

$s = \sin x \quad -1 \leq s \leq 1$

$\therefore |1+s| = 1+s, \ |1-s| = 1-s$

$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C$$

$$= \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C$$

Eg 7 $\int x\sqrt{2x+1}\,dx$

let $u=\sqrt{2x+1}$

$u^2=2x+1 \Rightarrow 2u\,du=2\,dx$

(i.e. $2u\frac{du}{dx}=2$)

$$=\int \overbrace{\frac{u^2-1}{2}}^{x}\;\overbrace{u}^{\sqrt{2x+1}}\cdot(\overbrace{u\,du}^{dx})$$

$$=\frac{u^5}{10}-\frac{u^3}{6}+C=\frac{(2x+1)^{\frac{5}{2}}}{10}-\frac{(2x+1)^{\frac{3}{2}}}{6}+C$$

<u>or</u> let $v=2x+1$, $dv=2\,dx$

$$=\int\frac{v-1}{2}\sqrt{v}\,\frac{dv}{2}=\frac{v^{\frac{5}{2}}}{10}-\frac{v^{\frac{3}{2}}}{6}+C$$

$=$ Same.

Eg 8 $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

$$= \int \frac{d(z^2+1)}{(z^2+1)^{\frac{1}{3}}}$$

$$= \int (z^2+1)^{-\frac{1}{3}}\, d(z^2+1)$$

$$= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C$$

Eg 9 $\int \sin^2 x\, dx$

$$= \int \frac{1-\cos 2x}{2}\, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Eg10 $\int \sin^3 x \, dx$

$= \int \sin^2 x \sin x \, dx$

$= -\int (1-\cos^2 x) \, d\cos x$

$= -\int (1-c^2) \, dc$

$= -c + \frac{c^3}{3} + C$

$= -\cos x + \frac{\cos^3 x}{3} + C$