Recall

$$(*)\ \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta x_k f(c_k) = \int_a^b f(x)\,dx$$

where $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

$$\Delta x_k = x_k - x_{k-1}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(x_k)$$

(if the limit $(*)$ exists)

= Signed area under graph of $y = f(x)$ between a and b

Eg 1 $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n}\sqrt{1-\left(\frac{k}{n}\right)^2}$

Sol write $\frac{1}{n} = x_k - x_{k-1}$

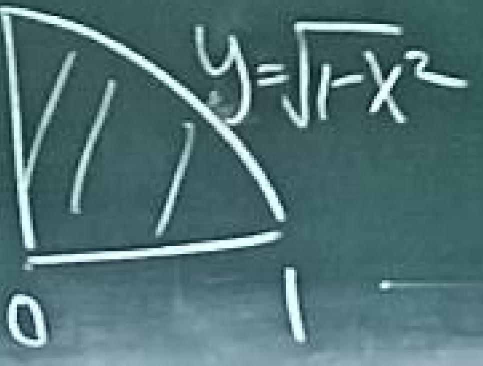
i.e. $x_k = \frac{k}{n}$

$\Rightarrow \mathbb{P} = \{0 = x_0 < x_1 \cdots < x_n = 1\}$

i.e. $[a, b] = [0, 1]$

$\sqrt{1-\left(\frac{k}{n}\right)^2} = \sqrt{1-x_k^2} = f(x_k)$

$\text{Ans} = \int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx$

$=$ 

$= \frac{\pi r^2}{4} = \frac{\pi}{4}$

Eg2 $\int_0^b x^2\,dx = ?$

Sol: $x_k = \frac{bk}{n}$

$\Delta x_k = \frac{b}{n}$

$$Ans = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b}{n}\left(\frac{bk}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{b^3}{3}$$

How about $\int_a^b x^2\,dx$?

Rm $a<b$, $\int_b^a f(x)\,dx = ?$

We define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

With this definition,
we always have

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

regardless of ordering of a, b, c.

i.e. $a<b<c$, $c<a<b$
$b<c<a$, $a<c<b$, etc.

Eg 3. $\int_a^b x^2\,dx = ?$

Sol: $= \int_0^b x^2\,dx - \int_0^a x^2\,dx$

$= \dfrac{b^3}{3} - \dfrac{a^3}{3}$

Similar limiting process can be used to evaluate

$\int_0^1 x^n\,dx,\ n = 0, 1, 2, \cdots$

$\int_0^1 \begin{matrix}\sin x \\ \cos x\end{matrix}\,dx$, (need trigonometric identities ...)

$\int_0^1 e^x\,dx$. (Sum of geometric series)

Fundamental Theorem of Calculus

Part I: If f is cont. on $[a,b]$ then $F(x) \stackrel{\text{def}}{=\!=} \int_a^x f(t)\,dt$ is cont. on $[a,b]$, differentiable on (a,b) and

$$F'(x) = f(x)$$

i.e. $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Part II:

If G is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

i.e. $G(b) - G(a) = \int_a^b G'(x)\,dx$.

Rm The textbook use F here.

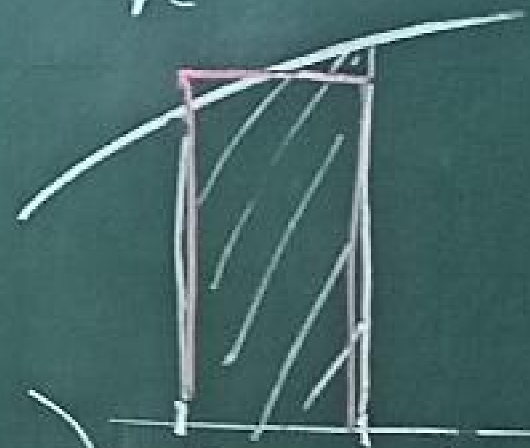
Rm $\int_a^b f(x)\,dx$: definite integral.

$\int f(x)\,dx \equiv \int^x f(t)\,dt$: antiderivative of f (indefinite integral)

Pf (part I)

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_a^{x+h}(*) - \int_a^x(*)}{h}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$



$$= \Big(\underbrace{\text{average of } f \text{ on } [x, x+h]}_{(**)}\Big)$$

$$\Rightarrow \min_{[x,x+h]} f \le (**) \le \max_{[x,x+h]} f$$

$$\xrightarrow{\text{Thm 3}} (**) = f(c),\ c \in [x, x+h]$$

$$\therefore \frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

Part (II) proof:

Let $F(x) = \int_a^x f(t)\,dt$

$$\xRightarrow{\text{Part I}} F'(x) = f(x) = G'(x)$$

$$(F-G)' = 0 \Rightarrow G(x) = F(x) + C$$

$$\therefore G(b) - G(a) = \left(F(b) + C\right) - \left(F(a) + C\right)$$

$$= F(b) - F(a) = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0}$$

$$= \int_a^b f(x)\,dx$$

Examples for Part (I)

Eg 4 $\frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1$

Eg 5 $\frac{d}{dx}\int_x^0 \sin(t^2)\,dt$

$= \frac{d}{dx}\left(-\int_0^x \sin(t^2)\,dt\right)$

$= -\sin(x^2)$

Eg 6 $\frac{d}{dx}\int_1^{x^2} e^{\frac{1}{t}}\,dt = \frac{d}{dx}F(x^2)$

$= \frac{d}{du}F(u)\Big|_{u=x^2} \cdot \frac{d}{dx}x^2$

$= e^{\frac{1}{x^2}} \cdot 2x$

Eg 7 $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \frac{1}{t^2+1}\,dt$

$$= \frac{d}{dx}\left(\int_0^{x^3} - \int_0^{x^2}\right)$$

$$= \left(\frac{1}{t^2+1}\right)_{t=x^3} \cdot 3x^2 - \left(\frac{1}{t^2+1}\right)_{t=x^2} \cdot 2x$$

$$= \frac{3x^2}{x^6+1} - \frac{2x}{x^4+1}$$

Examples for part II

$$\int_a^b G'(x)\,dx = G(b) - G(a) = G(x)\Big|_a^b$$

Eg 8: $\displaystyle\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi}$

$$= \sin\pi - \sin 0 = 0$$

Eg 9 $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$

$\displaystyle = \int_1^4 \left(\frac{3}{2}x^{\frac{1}{2}} - 4x^{-2}\right)dx$

$\displaystyle = \left(x^{\frac{3}{2}} + 4x^{-1}\right)\Big|_1^4$

$= 9 - 5 = 4$

Eg 10 $\displaystyle\int_0^1 \frac{1}{x^2+1}\,dx$

$\displaystyle = \tan^{-1}x \Big|_0^1 = \frac{\pi}{4} - 0$

Eg 11 $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx =$ 

$\displaystyle = \frac{\pi}{2}$