Antiderivative

Def: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Rm: If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$. for any constant C.

Def: $\int f(x)\,dx$ = the collection of all $F(x)+C$, such that $F'(x)=f(x)$

i.e. = the collection of all antiderivatives of $f(x)$

Eg 1 $f(x)$ $\int f(x)\,dx$

$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \dfrac{x^{n+1}}{n+1}+C, & n\neq -1 \\ \ln\lvert x\rvert + C, & n=-1 \end{cases}$
$\sin kx$	$\frac{-1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
---- etc.	
e^{kx}	$\frac{1}{k}e^{kx}+C$
a^{kx}	$\frac{1}{k\ln a}a^{kx}+C$

$(=(e^{\ln a})^{kx} = e^{k\ln a\, x}$

Eg2 Solve $\begin{cases} \dfrac{dv}{dt} = -32 \\ v(0) = 12 \end{cases}$

$v(t) = ?$

Sol

$$v(t) = -32t + C$$

$$v(0) = 12$$

$$\Rightarrow C = 12$$

$$\Rightarrow v(t) = -32t + 12$$

Eg 3 Solve $\begin{cases} \dfrac{d^2x}{dt^2} = -9.8 \\ x(0) = 10. \\ x'(0) = 0 \end{cases}$

Sol

$$\frac{d^2x}{dt^2} = -9.8 \Rightarrow \frac{dx}{dt} = -9.8t + C_1$$

$$\Rightarrow x(t) = -4.9t^2 + C_1 t + C_2$$

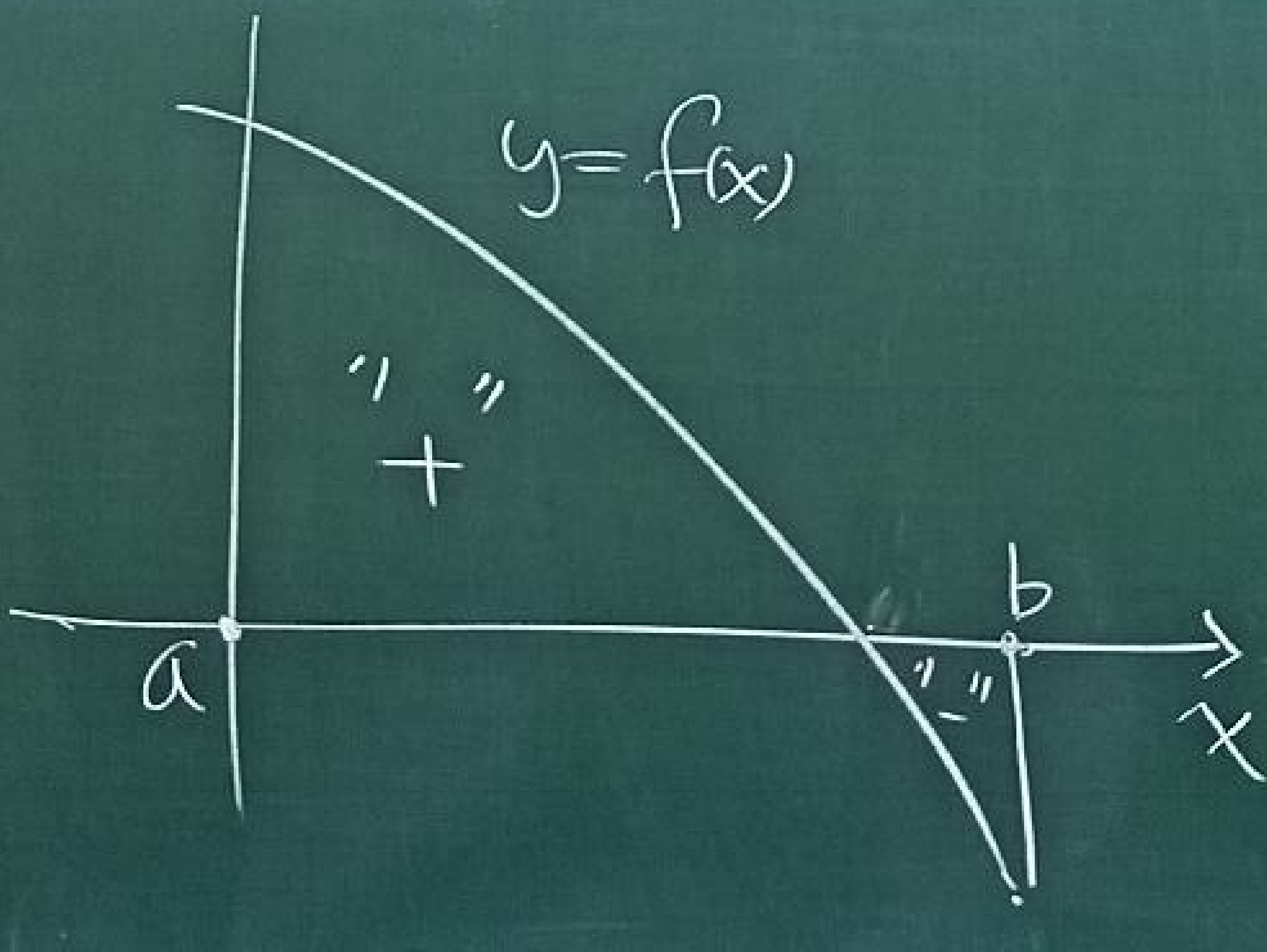
$$\begin{cases} x(0) = 10 \\ x'(0) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 10 \\ C_1 = 0 \end{cases}$$

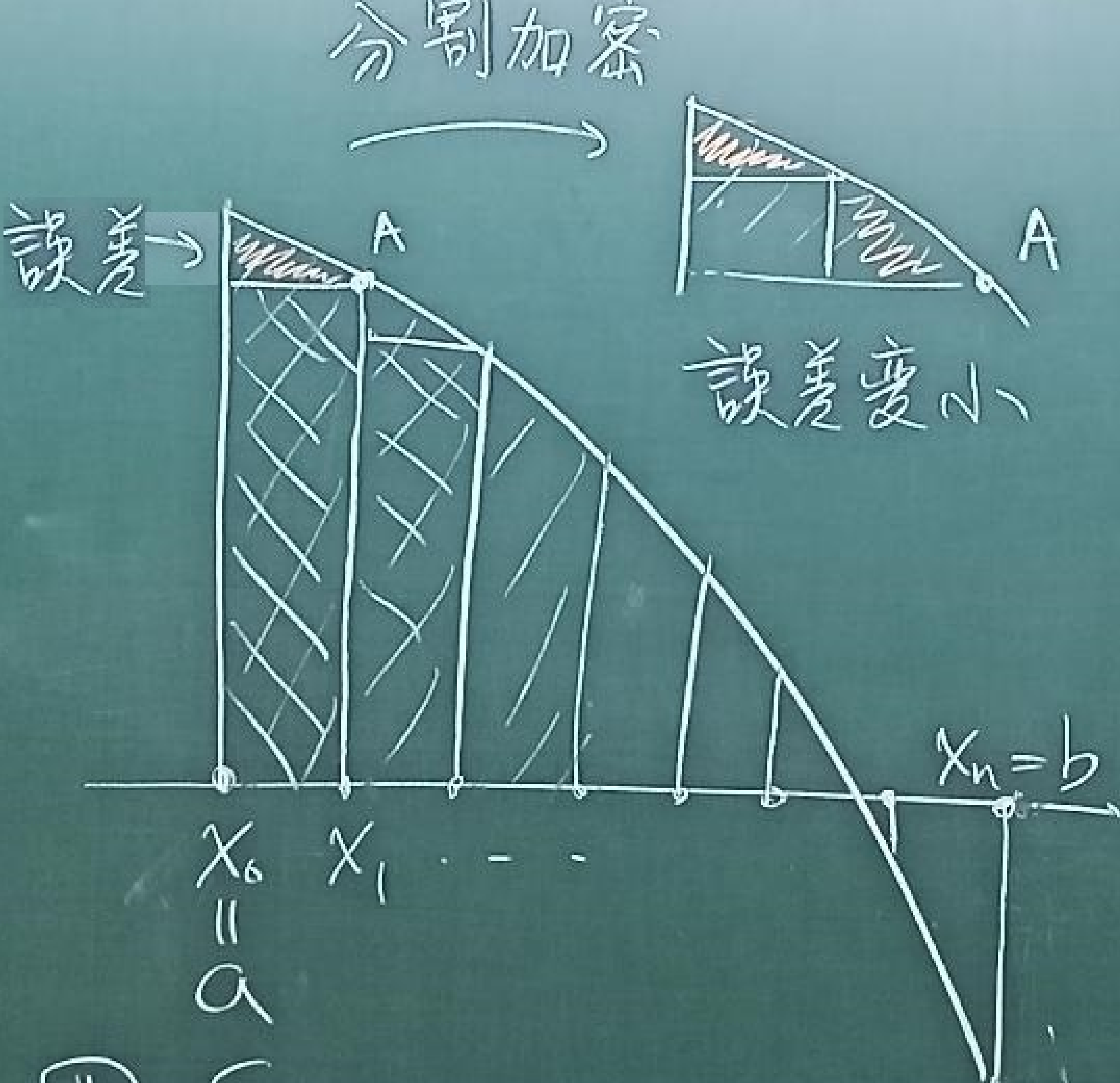
$$\therefore x(t) = -4.9t^2 + 10$$

Exercise: What if $\begin{cases} x(0) = 10 \\ x(1) = 0 \end{cases}$?

Integration

How to compute the "signed area" under the graph of $f(x)$ on $[a,b]$





$$\mathbb{P}=\{a=x_0, x_1, x_2, \cdots x_n=b\}$$

$$a=x_0<x_1<x_2 \cdots < x_n=b$$

For example, let $x_{i+1}=x_i+\frac{b-a}{n}$
(uniform partition)

We approximate the signed area by

$$A_n = \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$$

where $x_{k-1} \leq c_k \leq x_k$

Here in the picture,

$c_k = x_k$.

If $x_k = x_{k-1} + \frac{b-a}{n}$, $c_k = x_k$

We formally obtained the ~~sign~~ signed area by $\lim_{n\to\infty} A_n$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k)$$

Definite Integral

$$a = x_0 < x_1 \cdots < x_n = b$$

$$\mathbb{P} = \{x_0, x_1 \ldots x_n\} \text{ (Partition)}$$

$$\|\mathbb{P}\| \stackrel{\text{def}}{=\!=} \max_{1 \le k \le n} (x_k - x_{k-1})$$

= length of the largest interval

Signed area is defined as

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=\!=} \lim_{\|\mathbb{P}\| \to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$$

(where $c_k \in [x_{k-1}, x_k]$ is arbitrary)

provided the limit exist and we say f is integrable on [a, b]

Precise definition of

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) = I$$

For any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

$$\|P\| < \delta \Rightarrow \left| \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) - I \right| < \varepsilon$$

Thm: If f is cont. on $[a,b]$ (or only a few jump discontinuities) then this limit exist.

Rm Why use $\lim\limits_{\|P\| \to 0}$, not $\lim\limits_{n \to \infty}$ as definition?

Ans: There are some strange functions where $\lim\limits_{n \to \infty}$ exists, but $\lim\limits_{\|P\| \to 0}$ does not exist. We want to exclude these functions, and ~~not call them integrable~~ not consider them.

Eg: $f(x) = \begin{cases} 0 & x = \text{rational} \\ 1 & x = \text{irrational} \end{cases}$

Consider $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k)$

If $x_k - x_{k-1} = \frac{b-a}{n} = \frac{1}{n} \Rightarrow x_k = \frac{k}{n}$

$\therefore f(x_k) = 0,\ k = 1, \dots, n$

$\therefore \int_0^1 f(x)dx = 0$ (if we use $\lim_{n\to\infty}$)

However $\int_0^{\sqrt{2}-1} f(x)dx = \lim_{n\to\infty} (\cdots)$

$\Rightarrow x_k = \frac{k(\sqrt{2}-1)}{n},\ f(x_k) = 1$

$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k) = \sqrt{2}-1$

$f(x) \geq 0 \quad \int_0^1 f(x)dx = 0 < \sqrt{2}-1 = \int_0^{\sqrt{2}-1} f(x)dx$

since [0,1] is the larger interval, the result is not reasonable

This is why $\lim_{n\to\infty}$ is not a good definition for $\int_a^b f(x)dx$

By requiring the limit exist for arbitrary $\mathbb{P}$ and C_k.

we can exclude these strange functions from consideration

Rm f(x) is integrable on $[a,b]$

If $\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$

exists