Recall, for any $a, b, c > 0$

$$(\ln x)^a \ll x^b \ll e^{cx}$$

as $x \to +\infty$ 

and

$$(|\ln x|)^a \ll x^{-b} \ll e^{\frac{c}{x}}$$

as $x \to 0^+$ 

where $f(x) \ll g(x)$

means

$$\lim \frac{f(x)}{g(x)} = 0$$

Eg 1: $\lim_{x\to 0^+} x^x$ "0^0"

Sol: $= \lim_{x\to 0^+} \left(e^{\ln x}\right)^x$

$= \lim_{x\to 0^+} e^{x \ln x}$ "$e^{-0\cdot\infty}$"

$= \lim_{x\to 0^+} e^{\frac{\ln x}{\frac{1}{x}}}$ "$e^{\frac{-\infty}{\infty}}$"

$= e^{\lim_{x\to 0^+}\left(\frac{\ln x}{x^{-1}}\right)}$

$= e^{\lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}} = e^{\lim_{x\to 0^+}(-x)} = 1$

Proof of l'Hôpital's Rule

(the $\lim_{x\to a}$ "$\frac{0}{0}$" version)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

MVT?

$$\overset{(*)}{=} \lim_{x\to a}\frac{f'(c)}{g'(c)}, \text{ for some } c \text{ between } a, x$$

$$\overset{(**)}{=} \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad \left(\text{if this limit } = \begin{cases} L \\ +\infty \\ -\infty \end{cases}\right)$$

MVT $\not\Rightarrow$ (*) Since the "c" may be different for f and g

Cauchy's Mean Value Theorem ($\Rightarrow$ (*))

If f, g are cont. on $[a,b]$ and diff. on (a,b), Then ther exists $c \in (a,b)$ such that

$$\begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix} = 0$$

Pf: Define $F(x) = \begin{vmatrix} f(b)-f(a) & f(x)-f(a) \\ g(b)-g(a) & g(x)-g(a) \end{vmatrix}$

$F(a)=0, \quad F(b)=0$

$\overset{MVT}{\Longrightarrow} \exists\, c \quad 0 = F'(c) = \begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix}$ #

Rm From the proof of l'Hôpital's Rule, it is clear that if

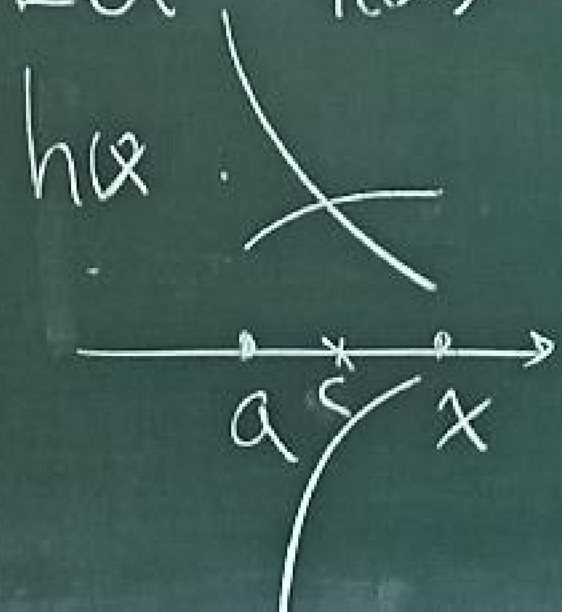
$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

then

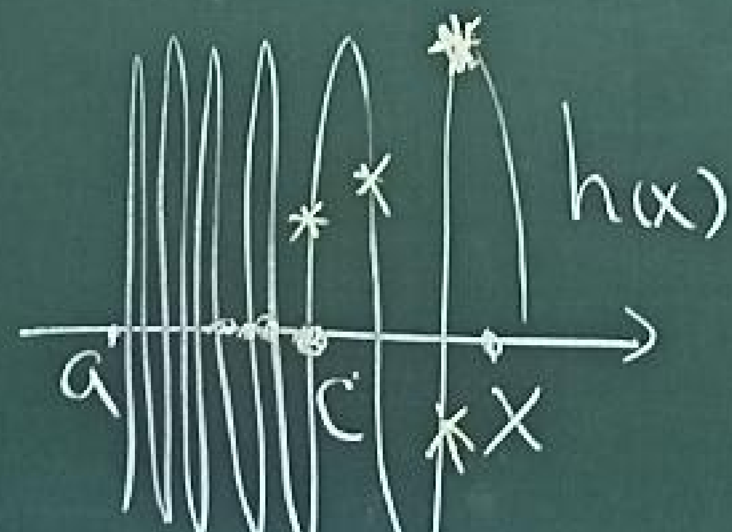
$$\lim_{x\to a}\frac{f'(c)}{g'(c)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad (**)$$

for any c between a and x

Let $h(x) = \dfrac{f'(x)}{g'(x)}$



$$\begin{cases} L \\ +\infty \\ -\infty \end{cases}$$



However, if $\lim_{x\to a} h(x)$ is oscillatory (or does not exist)

and c is any point between a and x

It is possible that

$$\lim_{x\to a} \frac{f'(c)}{g'(c)} \text{ converges.}$$

Eg $h(x) = \sin\left(\frac{1}{x}\right)$

c between 0 and x

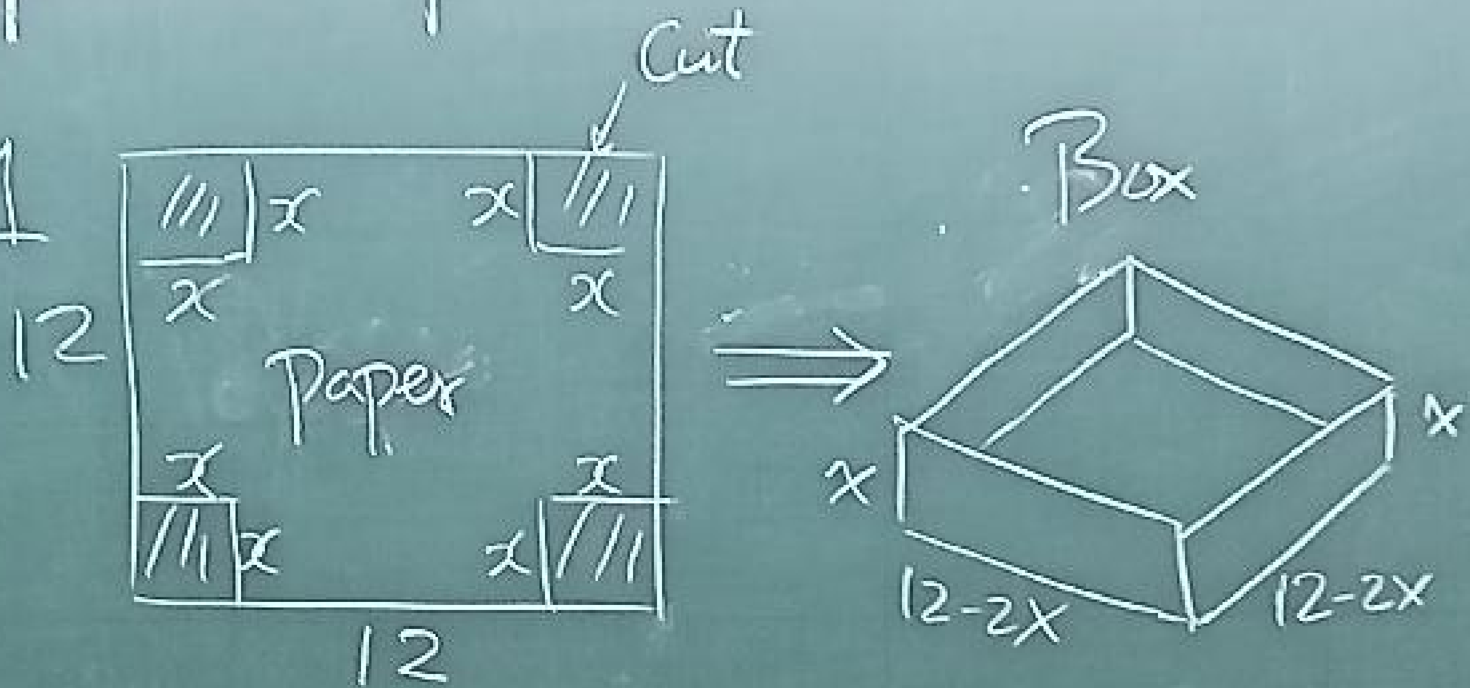
with $\sin\left(\frac{1}{c}\right) = 0$

i.e.

lim f'(x)/g'(x) DNE ≠> lim f(x)/g(x) DNE

Applied Optimization

Eg1



Find maximal volume of the box.

Sol:

$$V(x) = (12-2x)^2 \cdot x$$

find global max on $V(x)$ on $\overline{0 \le x \le 6}$

$$V'(x) = 12x^2 - 96x + 144 = 12(x-2)(x-6)$$

$V'(x)$: $+$ | $-$ | $+$

x: 0, 2, 6

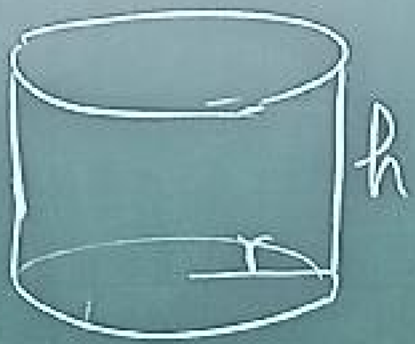
$V(x)$: $\nearrow$ 2 $\searrow$ 6 $\nearrow$

local min: $f(0), f(6)$, local max: $f(2)$

Compare values of f on critical point $(x=2)$ and boundary $(x=0, 6)$

$\Rightarrow$ global max $= f(2)$

Eg2.



Fix $V = \pi r^2 h = 1000$
find minimal surface area of the cylinder.

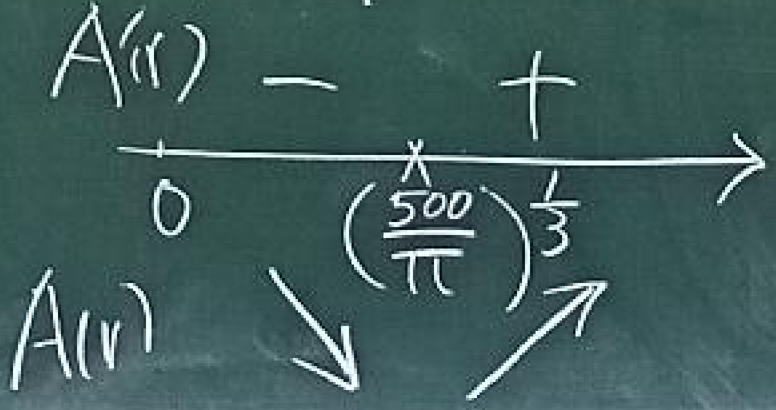
Sol: $A = 2\cdot\pi r^2 + 2\pi r\cdot h$

$= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$

$\therefore A(r) = 2\left(\pi r^2 + \frac{1000}{r}\right)$ on $r > 0$

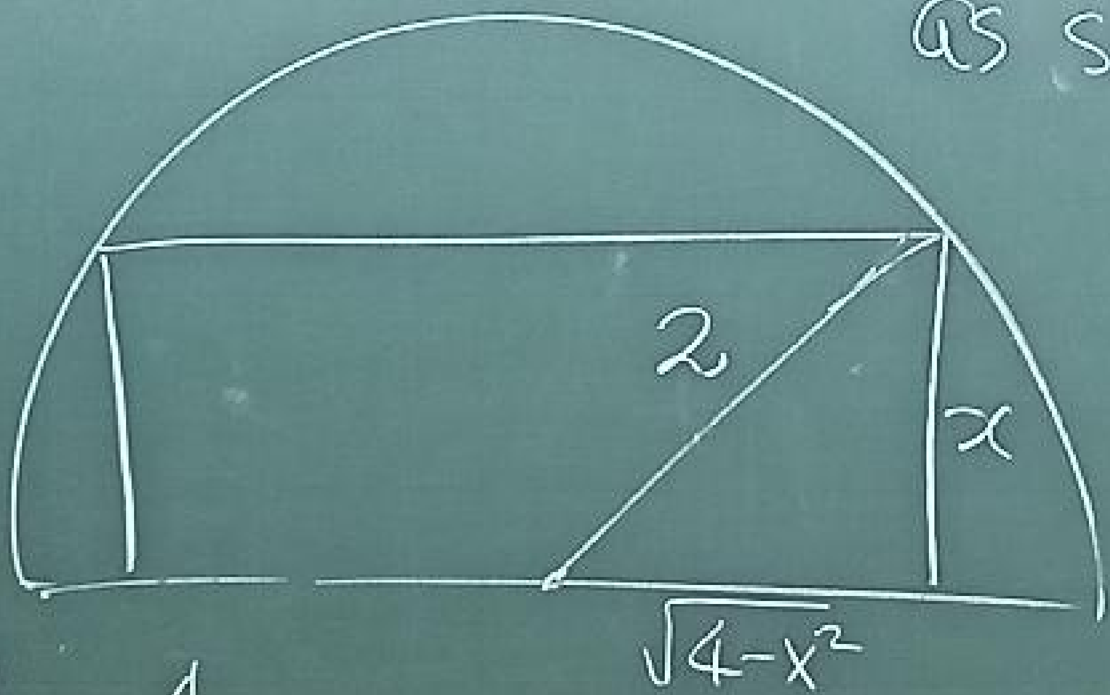
$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi\left(r^3 - \frac{500}{\pi}\right)}{r^2}$

r^2 always > 0, $r^3 > \frac{500}{\pi} \Leftrightarrow r > \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$



global min
$= A\left(\left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right)$

Eg 3: Maximize the area of the rectangle as shown below.



<u>Ans</u>: Global max $= A(\sqrt{2}) = 4$

<u>Sol</u>: $A(x) = 2\sqrt{4-x^2}\cdot x \geq 0$

Maximize $A(x)$ on $0 \leq x \leq 2$

<u>Trick</u>: Maximize $B = A^2(x) = 4(4-x^2)x^2$ on $0 \leq x \leq 2$

$= $ Maximize $4(4-y)y$ on $0 \leq y \leq 4$

$\frac{dB}{dy} = 8\cdot(2-y)$

$\frac{dB}{dy}$		+		−	
y	0		2		4
B	0	$\nearrow$		$\searrow$	

$\Rightarrow$ Global max $\Leftrightarrow y = 2 \Leftrightarrow x = \sqrt{2}$

Eg 4. x : # of product, $\underline{x \geq 0}$

$r(x) = 9x$: revenue

$c(x) = x^3 - 6x^2 + 15x$: cost

Find maximal profit

$$p(x) = r(x) - c(x) = -x^3 + 6x^2 - 6x$$
$$= -x(x^2 + 6x - 6)$$

$\underline{\text{Sol}}$. $$p'(x) = -3x^2 + 12x - 6$$
$$= -3(x - x_-)(x - x_+)$$

where $x_{\pm} = 2 \pm \sqrt{2}$

	0		x_-		x_+	
$p'(x)$		$-$		$+$		$-$
$p(x)$		$\searrow$		$\nearrow$		$\searrow$

local max

$p(0)$, $p(x_+)$

Check $p(0) = 0$

$p(x_+) \underset{?}{\geq} 0$

Which is globle max? $p(0)$ or $p(x_+)$?

Is $P(x_+)$ positive?

Ans: $P(x) = -x(x^2+6x-6)$

$x^2+6x-6 = (x+3)^2-15$

has 2 real roots

$\therefore$ $P(x)$ has 3 real roots

$\therefore$ $P(x_+)>0 =$ global max