Eg1 $\lim_{x\to 0} \frac{x-\sin x}{x^3} = ?$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

Sol. $= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$= \lim_{x\to 0} \frac{\sin x}{6x}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$= \lim_{x\to 0} \frac{\cos x}{6}$ (not "$\frac{0}{0}$", stop!)

$= \frac{1}{6}$

Eg2 $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2} = ?$ $\left(\text{"}\dfrac{0}{0}\text{"}\right)$

Sol: $= \lim\limits_{x\to 0} \dfrac{\sin x}{1+2x}$ $\left(\begin{matrix}\text{not "}\frac{0}{0}\text{"} \\ \text{, stop}\end{matrix}\right)$

$\neq \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$

incorrect

$= \dfrac{0}{1}$

correct

$\frac{\infty}{\infty}$ version of l'Hôpital's Rule.

If $\displaystyle\lim_{x\to a,\, a^+,\, a^-,\, \pm\infty} f(x) = \pm\infty$

and $\displaystyle\lim_{x\to a,\, a^+,\, a^-,\, \pm\infty} g(x) = \pm\infty$

and $\displaystyle\lim_{x\to a,\, a^+,\, a^-,\, \pm\infty} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$

Then $\displaystyle\lim_{x\to a,\, a^+,\, a^-,\, \pm\infty} \frac{f(x)}{g(x)} = \lim_{x\to a,\, a^+,\, a^-,\, \pm\infty} \frac{f'(x)}{g'(x)}$

Eg 3 $\lim_{x \to \infty} x \sin\frac{1}{x} = ?$ $(\infty \cdot 0)$

$$= \lim_{x \to \infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \quad \left(\text{"}\frac{0}{0}\text{"}\right)$$

$$= \lim_{x \to \infty} \frac{\cancel{-\frac{1}{x^2}} \cos\left(\frac{1}{x}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = 1$$

Eg4 $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{\frac{1}{2}}} = ?$ $\quad$ "$\dfrac{\infty}{\infty}$"

Sol: $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{2}x^{\frac{-1}{2}}}$ $\left(\text{"}\dfrac{0}{\infty} = 0\text{"}\right)$

$= \lim\limits_{x\to\infty} 2x^{\frac{-1}{2}} = 0$

Same result for $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{k}}$ for any $k > 0$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{kx^{k-1}} = \lim\limits_{x\to\infty} \dfrac{1}{kx^{k}} = 0$

Rm $a, b, c > 0$

$$\lim_{x \to \infty} \frac{e^{ax}}{x^b} = \left(\lim_{x \to \infty} \left(\frac{e^{\frac{a}{b}x}}{x} \right) \right)^b = \infty$$

$$\lim_{x \to \infty} \frac{(\ln x)^c}{x^b} = \left(\lim_{x \to \infty} \frac{\ln x}{x^{\frac{b}{c}}} \right)^c = 0$$

In this sense, we write

$$e^{ax} \gg x^b \gg (\ln x)^c$$

as $x \to \infty$

for any $a, b, c > 0$

Eg 5 $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = ? \left(\text{"}\frac{0}{0}\text{"}\right)$

$\left(\text{Note } \lim_{x\to 0^+} e^{-\frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = \frac{1}{e^{\infty}} = 0\right)$

Sol: $= \lim_{x\to 0^+} \frac{\frac{1}{x^2} e^{-\frac{1}{x}}}{1} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^2} \left(\text{"}\frac{0}{0}\text{"}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{x^2} e^{-\frac{1}{x}}}{2x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{2x^3} \left(\text{"}\frac{0}{0}\text{"}\right)$

$= \cdots\cdots$ does not work

Instead, we write

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} \left(\text{"}\frac{\infty}{\infty}\text{"}\right)$$

$$= \lim_{x\to 0^+} \frac{\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}\, e^{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = 0$$

<u>Rm</u> $\lim\limits_{x\to 0^+} \dfrac{e^{-\frac{1}{x}}}{x^k}$, $k>0$

$= 0$ (homework)

Eg 6 $\lim\limits_{x\to 0} \dfrac{x^2\cos\frac{1}{x}}{\sin x} = ?$ $\left(\text{"}\dfrac{0}{0}\text{"}\right)$

$$\overset{?}{=} \lim_{x\to 0} \frac{2x\cos\frac{1}{x} + x^2\left(\frac{-1}{x^2}\right)\sin\frac{1}{x}}{\cos x}$$

$$= \lim_{x\to 0} \frac{2x\cos\frac{1}{x} - \sin\left(\frac{1}{x}\right)}{\cos x} \quad \text{does not exist}$$

$$\overset{?}{\Rightarrow} \lim_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\sin x} \quad \text{does not exist}$$

<u>Ans</u>: l'Hôpital Rule only applies to $\lim \dfrac{f(x)}{g(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$. It does not apply in this case.

Instead, we write

$$\lim_{x \to 0} \frac{x^2 \cos\frac{1}{x}}{\sin x}$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)\left(x \cos\frac{1}{x}\right)$$

$$= \left(\lim_{x \to 0} \frac{x}{\sin x}\right)\left(\lim_{x \to 0} x \cos\frac{1}{x}\right)$$

$$= 1 \cdot 0 = 0$$

Other applications

"$\infty - \infty$", "1^{∞}", "0^0", "∞^0"

Eg7 $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ ("$\infty-\infty$")

$= \lim_{x\to 0}\frac{x-\sin x}{x\sin x}$ ("$\frac{0}{0}$")

$= \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}$ ("$\frac{0}{0}$")

$= \lim_{x\to 0}\frac{\sin x}{\cos x+\cos x-x\sin x} = \frac{0}{2}$

$= 0$

Eg 8 $\lim_{x\to\infty} x^{\frac{1}{x}}$ "∞^0"

$$= \lim_{x\to\infty} \left(e^{\ln x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \quad \left(e^{\text{"}\frac{\infty}{\infty}\text{"}}\right)$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = 1$$

Rm For any $a, b, c > 0$

$$(|\ln x|)^a \ll x^{-b} \ll e^{\frac{c}{x}}$$

as $x \to 0^+$