Eg1 Sketch the graph of

$$f(x) = x^4 - 4x^3 + 10$$

and mark all local extrema and points of inflection.

Sol

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

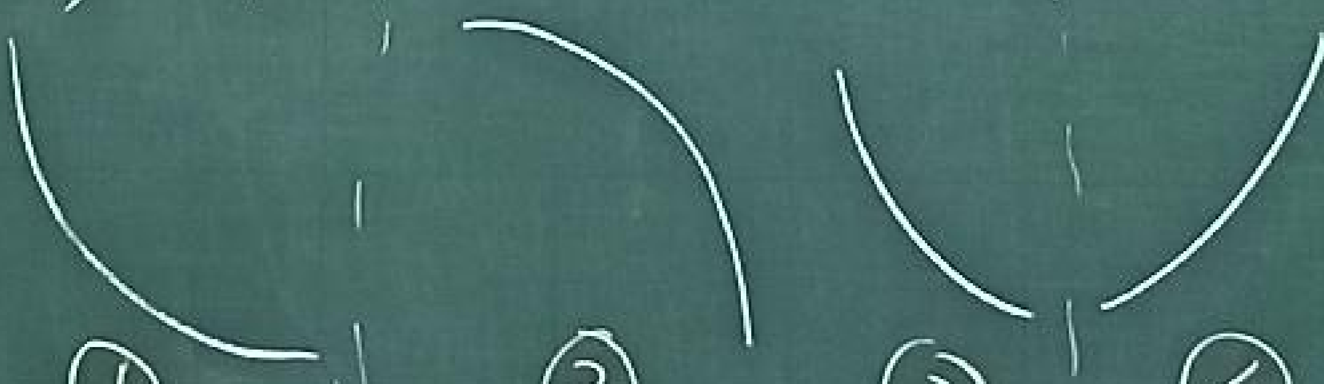
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

$f'(x) = 0$. $x = 0, 3$ $\quad f(0) = 10$

$f''(x) = 0$, $x = 0, 2$ $\quad f(2) = -6$

$f(3) = -17$

x^2	$+$		$+$		$+$		$+$
$x-3$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$-$		$-$		$+$
		0		2		3	
x	$-$		$+$		$+$		$+$
$x-2$	$-$		$-$		$+$		$+$
$f''(x)$	$+$		$-$		$+$		$+$



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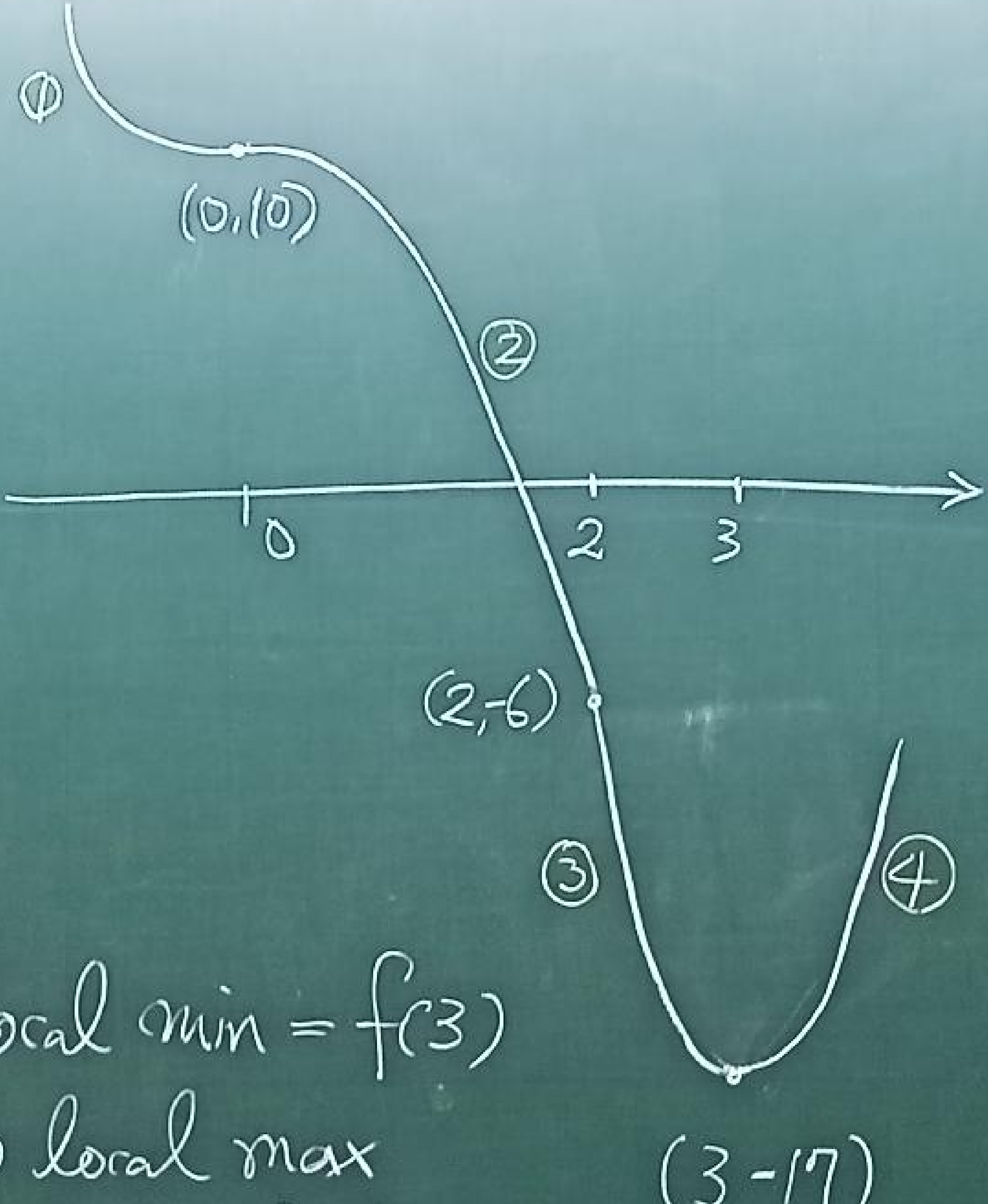


local min $= f(3)$

no local max

point of inflection

$(0, f(0)), (2, -6)$

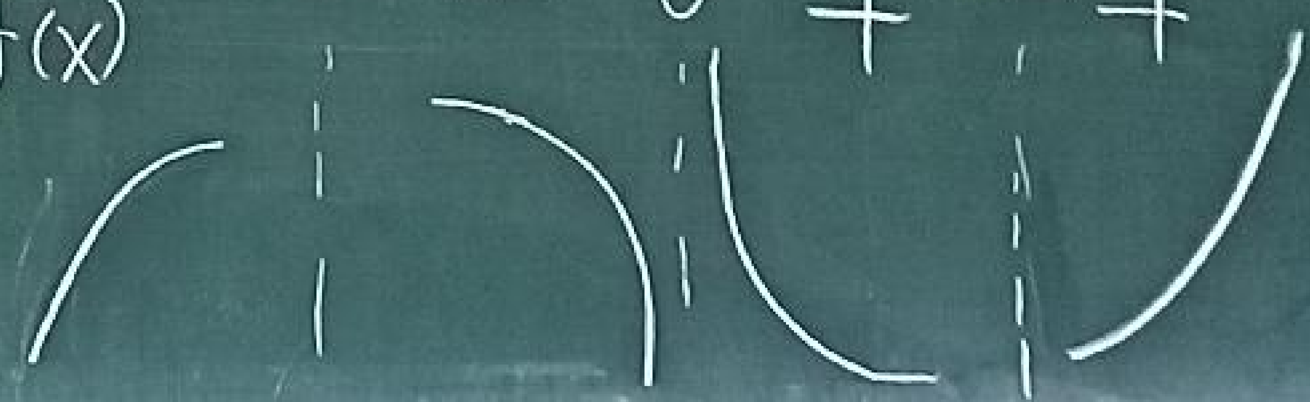
Eg 2 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$

Sol $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{(x-2)(x+2)}{2x^2}$$

$$f''(x) = 4x^{-3}$$

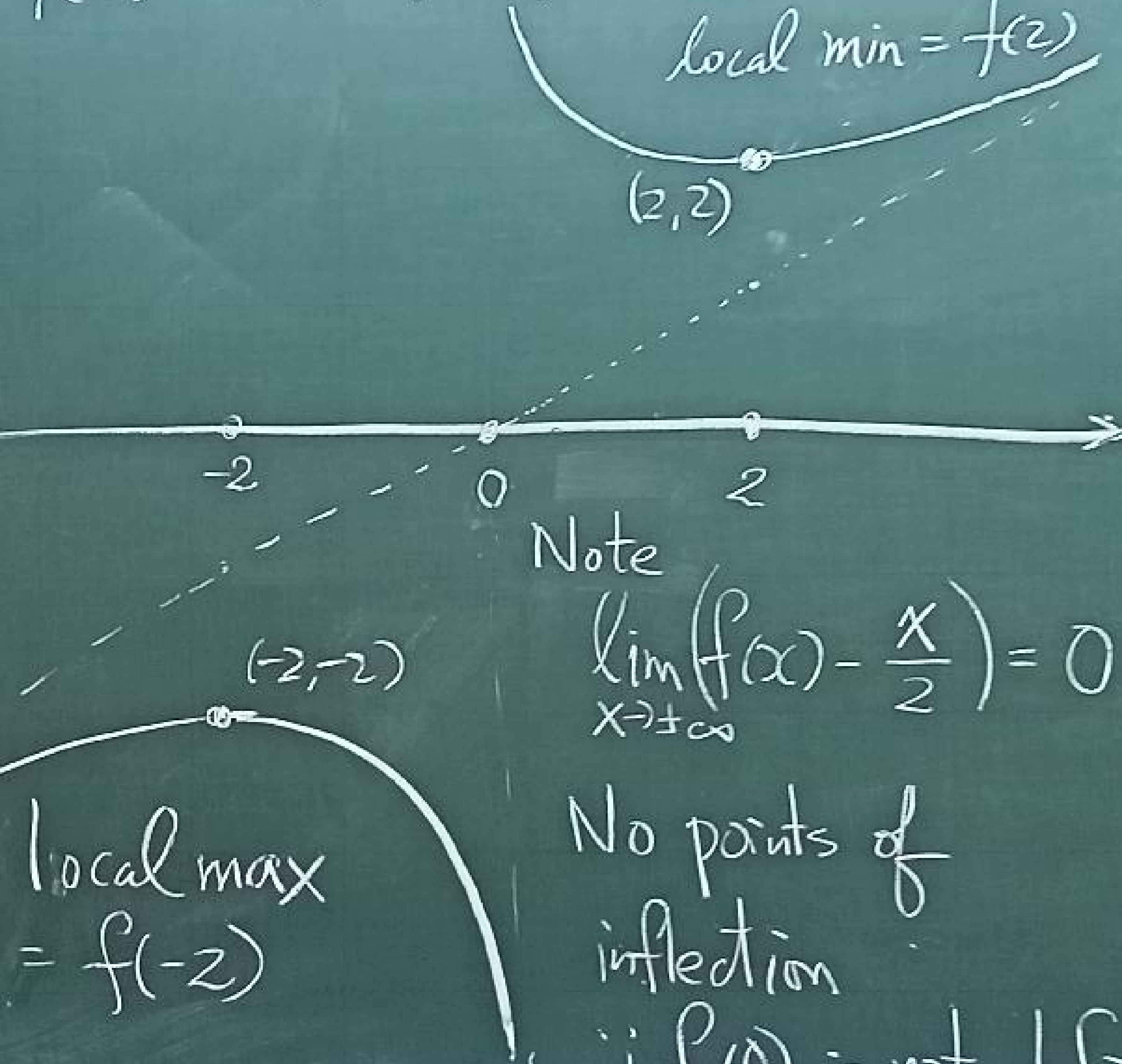
x^2	$+$		$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$x+2$	$-$		$+$		$+$		$+$
$f'(x)$	$+$		$-$		$-$		$+$
		-2		0		2	
$f''(x)$	$-$		$-$		$+$		$+$



$f(0) = \pm\infty$,

$f(2) = 2$, $f(-2) = -2$

local min $= f(2)$



Note

$$\lim_{x \to \pm\infty} \left(f(x) - \frac{x}{2} \right) = 0$$

local max
$= f(-2)$

No points of inflection

$\because f(0)$ is not defined

Eg3 Sketch the graph of

$$f(x) = x^{\frac{1}{3}}(x-4)$$

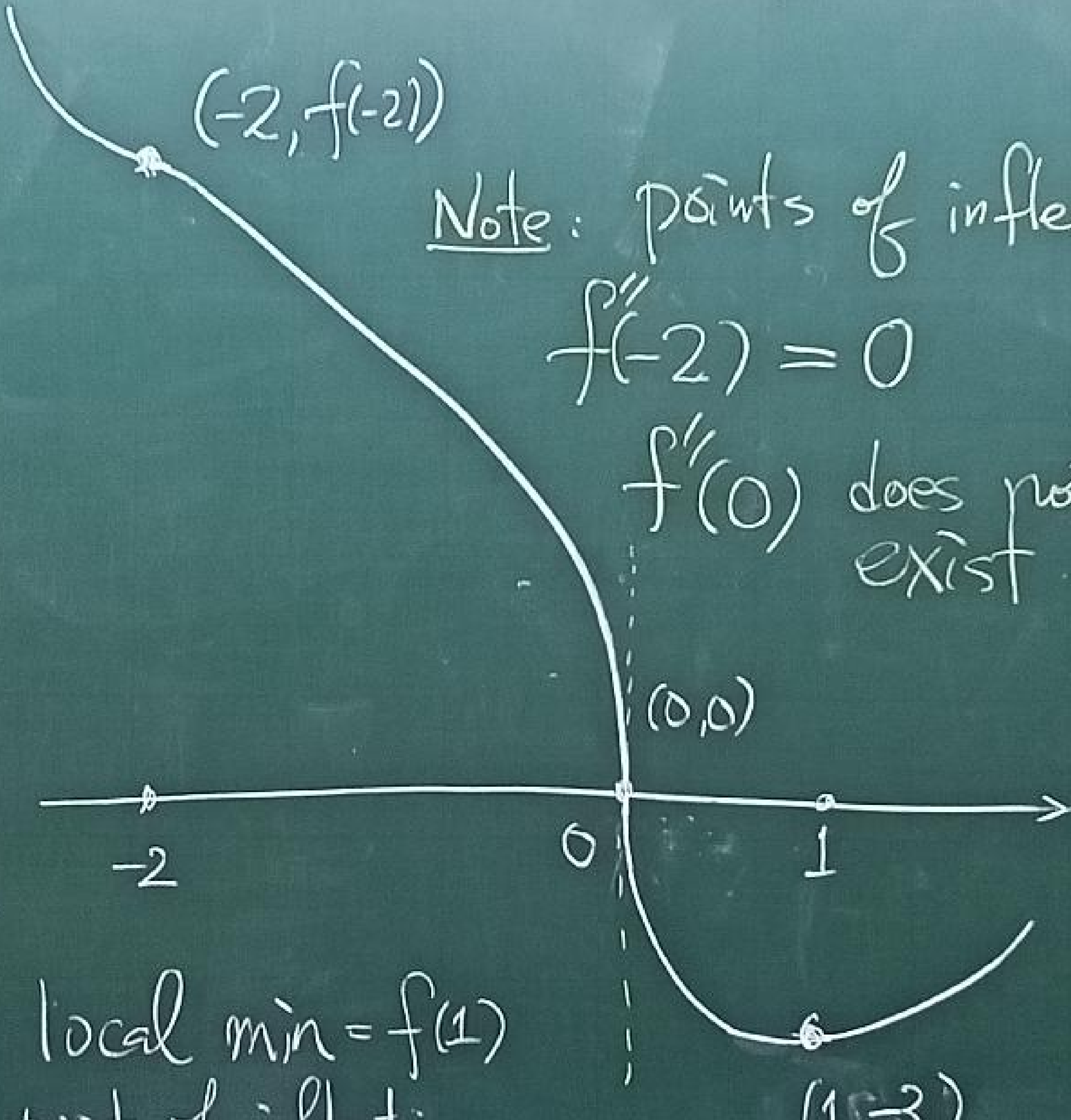
Sol $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{\frac{-2}{3}} = \frac{4}{3}x^{\frac{-2}{3}}(x-1)$$

$$f''(x) = \frac{4}{9}x^{\frac{-2}{3}} + \frac{8}{9}x^{\frac{-5}{3}} = \frac{4}{9}x^{\frac{-5}{3}}(x+2)$$

		-2		0		1	
$x^{\frac{2}{3}}$	+		+		+		+
$x-1$	−		−		−		+
$f'(x)$	−		−		−		+
$x^{\frac{5}{3}}$	−		−		+		+
$x+2$	−		+		+		+
$f''(x)$	+		−		+		+

$f(0)=0$, $f(1)=-3$, $f(-2)=6\sqrt[3]{2}$



<u>Note</u>: points of inflection

$f''(-2)=0$

$f''(0)$ does not exist.

local min $= f(1)$

point of inflection

$(-2, f(-2))$, $(0, f(0))$

L'Hôpital's Rule

How to evaluate "$\lim \frac{0}{0}$" and "$\lim \frac{\infty}{\infty}$"

Eg: $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

<u>Thm</u> If f and g are differentiable on $(a-\delta, a+\delta)$, $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \neq a$

If $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$

If the limit does not exist (eg: oscillatory), then l'Hopital's rule does not apply.

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

Eg1 $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta}$

$$\lim_{\theta \to \theta} \frac{(\sin\theta)'}{\theta'} = \lim_{\theta \to 0} \frac{\cos\theta}{1} = 1$$

$$\therefore \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0} \frac{\cos\theta}{1} = 1$$

Eg2 $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} = ?$

Sol $= \lim\limits_{\theta \to 0} \frac{\sin\theta}{2\theta} = \lim\limits_{\theta \to 0} \frac{\cos\theta}{2} = \frac{1}{2}$

Eg3 $\lim\limits_{x \to 0} (1 + ax)^{\frac{1}{x}}$ $\left(\text{"}1^{\infty}\text{"}\right)$

$$= \lim_{x \to 0} \left(e^{\ln(1+ax)}\right)^{\frac{1}{x}} = e^{\lim\limits_{x \to 0} \frac{\ln(1+ax)}{x}}$$

$$= e^{\left(\lim\limits_{x \to 0} \frac{a}{1+ax}\right)} = e^{a}$$

Rm l'Hôpital's Rule also applies

to $\lim\limits_{\substack{x \to a^{\pm} \\ x \to \pm\infty \\ x \to a}}$ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Eg 4 $\lim\limits_{x \to \infty} \frac{x^{4.5}}{e^{\frac{x}{100}}}$ (Similarly for any $\lim\limits_{x \to \infty} \frac{x^M}{e^{\varepsilon x}}$ $M > 0$ $\varepsilon > 0$)

Sol: $= \lim\limits_{x \to \infty} \frac{4.5\, x^{3.5}}{\frac{1}{100} e^{\frac{x}{100}}}$ (still "$\frac{\infty}{\infty}$")

$= \lim\limits_{x \to \infty} \frac{4.5 \cdot 3.5\, x^{2.5}}{\frac{1}{100} \frac{1}{100} e^{\frac{x}{100}}}$

$= \cdots = \lim\limits_{x \to \infty} \frac{\cdots\, x^{-0.5}}{\cdots\, e^{\frac{x}{100}}} = 0$