# Corollaries of M.V.T.

Cor. 1 If $f'(x) \equiv 0$ on $(a,b)$
then $f(x) = \text{constant}$ on $(a,b)$

pf: If $a < x_1 < x_2 < b$
and $f(x_1) \neq f(x_2)$.

$$\xrightarrow{\text{M.V.T}} \exists\, c \in (x_1, x_2) \text{ such that}$$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0, \text{ contradiction}$$

Cor. 2 If $f'(x) \equiv g'(x)$ on $(a,b)$,
then $f(x) = g(x) + C$ on $(a,b)$

<u>Cor 3</u>: Suppose $f$ is cont. on $[a,b]$ and differentiable on $(a,b)$.

If $f'(x) \gtrless 0$ for every $x \in (a,b)$

then $f(x)$ is $\begin{matrix}\text{increasing} \\ \text{decreasing}\end{matrix}$ on $[a,b]$

$\left( \begin{matrix}\text{increasing} \\ \text{decreasing}\end{matrix} : \ x_1 < x_2 \Rightarrow f(x_1) \lesseqgtr f(x_2) \right)$

<u>Pf</u>: (for the "$>0 \Rightarrow$ increasing" case)

If not true ("$>0$" and <u>not</u> <u>increasing</u>)

$\Rightarrow \exists x_1, x_2, \ a \le x_1 < x_2 \le b \ \ f(x_1) \ge f(x_2)$

$\xrightarrow{MVT} \exists c \in (x_1, x_2). \ f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \le 0$. contradiction

# First derivative test for local extrema

Suppose $f$ is cont. on $[a,b]$,

$c$ is a critical point, and

$f$ is differentiable on $(a,c)\cup(c,b)$

Then

| | | | $c$ | | |
|---|---|---|---|---|---|
| (i) | $f'(x)$ | "+" | | "−" | $\Rightarrow$ local max |
| | $f(x)$ | $\nearrow$ | | $\searrow$ | |
| (ii) | $f'(x)$ | "−" | | "+" | $\Rightarrow$ local min |
| | $f(x)$ | $\searrow$ | | $\nearrow$ | |

| | | | $c$ | | |
|---|---|---|---|---|---|
| (iii) | $f'(x)$ | "+" | | "+" | $\Rightarrow$ not a local extrema |
| | $f(x)$ | $\nearrow$ | | $\nearrow$ | |
| | $f'(x)$ | "−" | | "−" | not a local extrema |
| | $f(x)$ | $\searrow$ | | $\searrow$ | |

<u>Remark</u> Second derivative test

Suppose $f''$ is cont. on $(c-\delta, c+\delta)$ $\delta > 0$

$\left(\begin{matrix}\text{I} \\ \text{II}\end{matrix}\right)$ If $f'(c) = 0$, $f''(c) \begin{matrix} > 0 \\ < 0 \end{matrix}$

then $f$ has a local $\begin{matrix}\text{min} \\ \text{max}\end{matrix}$ at $c$

(iii) If $f'(c) = 0$, $f''(c) = 0$

$\Longrightarrow$ No conclusion.

<u>Eg 1</u>. $f_1(x) = x^3$, $f_2(x) = -x^3$

$f_3(x) = x^4$, $f_4(x) = -x^4$

First Derivative Test $\Longrightarrow$ $f_1, f_2$: Not local extrema at $x=0$, $f_3$: $x=0$: local min, $f_4$: $x=0$: local max

Eg2. Find all critical points of $f(x)=x^{\frac{1}{3}}(x-4)$ and determine whether they are local $\begin{smallmatrix}\text{min}\\ \text{max}\end{smallmatrix}$ or neither.

<u>Sol</u>: $f(x)=x^{\frac{4}{3}}-4x^{\frac{1}{3}}$

$$f'(x)=\frac{4}{3}x^{\frac{1}{3}}-\frac{4}{3}x^{\frac{-2}{3}}=\frac{4}{3}x^{\frac{-2}{3}}(x-1)$$

Critical points: 0, 1

| | | 0 | | 1 | |
|---|---|---|---|---|---|
| $x^{\frac{-2}{3}}$ | $+$ | | $+$ | | $+$ |
| $(x-1)$ | $-$ | | $-$ | | $+$ |
| $f'(x)$ | $-$ | | $-$ | | $+$ |
| $f(x)$ | $\searrow$ | 0 | $\searrow$ | 1 | $\nearrow$ |
| | | Neither | | local min | |

# Concavity and curve sketching

$f'' \begin{matrix} > 0 \\ < 0 \end{matrix} \Rightarrow f' \text{ is } \begin{matrix} \text{inc.} \\ \text{dec} \end{matrix}$

concave up (upward)

concave down (downward)

concave up: upward

, concave down

downward

Def. $(c, f(c))$ is called a point of inflection of $f$ if $f$ changes concavity across $c$

i.e. $f$ is concave $\begin{matrix} \text{upward} \\ \text{downward} \end{matrix}$ on $\underbrace{\begin{pmatrix} x<c \\ x>c \end{pmatrix} \text{ or } \begin{pmatrix} x>c \\ x<c \end{pmatrix}}_{\text{near } x=c}$

Eg 3. Is $(0, f(0))$ point of inflection?

| | | | |
|---|---|---|---|
| $f(x) =$ | $x^{\frac{5}{3}}$ | $x^4$ | $x^3$ |
| | $0$ | $0$ | $0$ |
| $f''(x)$ | $-\ \times\ +$ | $+\ 0\ +$ | $-\ 0\ +$ |
| point of inflection | Yes. | No. | Yes |

$$f''(c)=0 \ \underset{\displaystyle \overset{\not\Longleftarrow}{(x^{\frac{5}{3}})}}{\overset{(x^4,\, x^3)}{\not\Longrightarrow}} \ (c, f(c)) \text{ is a point of inflection}$$

$x^{\frac{5}{3}}$