__Def__ Let $D \subseteq \mathbb{R}$ be an interval.

$f: D \longmapsto \mathbb{R}$, then f attains

~~$c \in D$ is an~~ absolute (local). $\begin{matrix}\max\\ \min\end{matrix}$

at $c \in D$

if $f(x) \begin{matrix}\leq\\ \geq\end{matrix} f(c)$

for all $x \in D$

$x \in (c-\delta, c+\delta) \cap D$

$\delta > 0$

When does f have abs. $\begin{matrix} \max \\ \min \end{matrix}$ on D?

Eg1. $f_1 : (\frac{-\pi}{2}, \frac{\pi}{2}) \longmapsto \mathbb{R}$

$$f_1(x) = \tan x$$

Eg2: $f_2 : (0, \infty) \longmapsto \mathbb{R}$

$$f_2(x) = \frac{1}{x}$$

Both f_1 and f_2 are continuous. But Neither has abs. $\begin{matrix} \max \\ \min \end{matrix}$

[The Extreme Value Theorem]

Thm1: (pf beyond this course)

If f is cont. on $[a,b]$

Then there exist

$x_m, x_M \in [a,b]$, such that

$f(x_m)=m$, $f(x_M)=M$

are abs. min and abs. max

i.e.

$m = f(x_m) \leq f(x) \leq f(x_M) = M$

for all $x \in [a,b]$

Thm 2. If

(i) $f : [a,b] \longmapsto \mathbb{R}$ is cont.

(ii) f has a local $\begin{smallmatrix}\text{min}\\\text{max}\end{smallmatrix}$ at $c \in (a,b)$

(iii) f is differentiable at c

$\Longrightarrow f'(c) = 0$

$\not\Longleftarrow$

Eg 3: $f : [-1, 1] \longmapsto \mathbb{R}$

$f(x) = x^3$, $f'(0) = 0$

but 0 is neither local $\begin{smallmatrix}\text{min}\\\text{max}\end{smallmatrix}$

Pf of Thm 2

If f has a local min at c and $f'(c)$ exists, then

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$f'(c) = \lim_{y \to c^-} \frac{f(y) - f(c)}{y - c} \leq 0$$

$(y < c < x)$

$\Rightarrow f'(c) = 0.$

Similarly for local max #

$f : [a,b] \longmapsto \mathbb{R}$ continuous

How do we find abs. $\begin{matrix}\text{max}\\ \text{min}\end{matrix}$?

<u>Note</u>: <u>Possible</u> abs. $\begin{matrix}\text{min}\\ \text{max}\end{matrix}$ include

critical points (背) $\begin{cases} \text{(i) } c \in (a,b),\ f'(c) = 0 \\ \text{(ii) } c \in (a,b),\ f'(c) \text{ does not exist} \end{cases}$

(iii) $c = a$ or $c = b$

<u>Ans</u>: Step 1: find all candidates in (i), (ii) (iii)

<u>Step 2</u>: Compare values of f

Eg4 Find abs. $\begin{matrix}\text{min}\\ \text{max}\end{matrix}$

for $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

<u>Sol</u>: Critical points (ii) & (iii)

$f'(x) = \frac{2}{3} x^{\frac{-1}{3}}$

$= 0$ or does not exist

$\Rightarrow x = 0$

Possible candidates

$x = 0, \ -2, \ 3$

$f(x) = 0, \ \sqrt[3]{4}, \ \sqrt[3]{9}$

abs. min　　　abs. max

Rolle's Thm.

Suppose $h(x)$ is cont. on $[a,b]$ diff. on (a,b) and $h(a)=h(b)$

Then there exists $c\in(a,b)$ such that $h'(c)=0$

Mean Value Thm:

If $f(x)$ is cont. on $[a,b]$ and diff. on (a,b), then there exists $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Obviously M.V.T. $\Rightarrow$ Rolle's

In fact, we also have " $\Leftarrow$ "

Pf: Suppose Rolle's Thm holds.

Let $h(x) = f(x) - g(x)$

where $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$\Rightarrow g(a) = f(a),\ g(b) = f(b)$

$\Rightarrow h(a) = 0,\ h(b) = 0$

$\xrightarrow{\text{Rolle's}} \exists \in (a,b)$, such that

$0 = h'(c) = f'(c) - g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$ ※

Pf of Rolle's Thm.

Suppose $h\begin{pmatrix} x_m \\ x_M \end{pmatrix} = \text{abs.} \begin{matrix} \text{min} \\ \text{max} \end{matrix}$

on $[a, b]$. x_m, x_M could be

(i) $c \in (a,b)$ $h'(c) = 0$

(ii) $c \in (a,b)$. $h'(c)$ does not exist

(iii) $c = a$ or b

(ii) is not possible since h is diff.

Case I: $x_m \in$ (i) or $x_M \in$ (i)

$\Rightarrow h'(x_m) = 0$ or $h'(x_M) = 0$.

$\Rightarrow c = x_m$ or x_M

Case II: $x_m = a, x_M = b$ or $x_m = b, x_M = a$

$h(a) = h(b) \Rightarrow h(x_m) = h(x_M)$

$\Rightarrow h(x) = \text{constant} \Rightarrow h'(c) = 0$ for all $c \in (a,b)$