Linearization (SKIP "differential")

Def: If $f(x)$ is differentiable at $x=a$, then the linear function

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of $f(x)$ centered at a (near a)
(linear approximation)

a = center of approximation.

$L(x)$ = (the) a linear function satisfying "$f(x) \cong L(x)$ near a"

Eg 1: Find linearization of $(1+x)^k$ near $x=0$, $(k\in\mathbb{R})$

<u>Ans</u>: $f(0)=1$

$f'(0)=k\cdot(1+0)^{k-1}=k$

$\therefore L(x)=1+kx$.

Eg 2: find an approximate value of $\sqrt{1.001}$

<u>Ans</u>: $(1+x)^k$, $k=\frac{1}{2}$, $x=0.01$

$$\therefore \sqrt{1.001} \cong 1+\frac{1}{2}\,0.001 = 1.0005$$

Eg3 $(7.97)^{\frac{1}{3}} \cong ?$

Ans $= \left(8-0.03\right)^{\frac{1}{3}}$

$= \left(8\left(1-\frac{0.03}{8}\right)\right)^{\frac{1}{3}}$

$\cong 2\left(1-\frac{1}{3}\cdot\frac{0.03}{8}\right)$

$= 1.9975$

Eg4. $\sin\left(\frac{\pi}{6}+0.001\right) \cong ?$

Ans: Let $f(x) = \sin(x)$

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\cdot\left(x-\frac{\pi}{6}\right)$

Ans $= L\left(\frac{\pi}{6}+0.001\right) = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\cdot 0.001$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot 0.001$

What does it mean by
"$f(x) \approxeq L(x)$ near a"?

$$f(x) = L(x) + \text{error}$$

<u>Prop</u>: If f is diff. at $x=a$

then $$\lim_{x-a} \frac{f(x)-L(x)}{x-a} = 0$$

<u>Pf</u>: $$= \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} - \lim_{x \to a} \frac{f'(a)(x-a)}{(x-a)}$$

$$= f'(a) - f'(a) = 0$$

In general, we say

$L(x)$ is linearization of $f(x)$ near $x=a$ if

(i) $L(x)$ is linear $\left(=f(a)+m(x-a)\right)$

(ii) $\displaystyle\lim_{x\to a}\frac{f(x)-L(x)}{x-a}=0$

Similarly, $f(x)$ and $g(x)$ are tangent to each other ~~near~~ at $x=a$ if (first order approximation)

$$\lim_{x\to a}\frac{f(x)-g(x)}{x-a}=0$$

In summary, if f is diff. at $x=a$, then

$$\lim_{x\to a} \frac{\overbrace{f(x)-L(x)}^{\text{"error"}}}{x-a} = 0$$

i.e. $$\lim_{x\to a} \frac{\text{error}}{x-a} = 0$$

$$\therefore \text{error} = \frac{\text{error}}{x-a} \cdot (x-a)$$

$$= \left(\underbrace{\begin{array}{c}\text{something that}\\ \text{goes to zero,}\\ \text{as } x\to a\end{array}}_{\equiv \varepsilon} \right)(x-a)$$

$\therefore f(x)$ is diff. at $x=a$

$$\Longleftrightarrow f(x) = L(x) + \varepsilon \cdot (x-a)$$

(證明)

where $\lim\limits_{x \to a} \varepsilon = 0$

$$\left(\text{i.e. } \frac{\Delta f}{\Delta x} = f'(a) + \varepsilon, \ \lim_{\Delta x \to 0} \varepsilon = 0 \right)$$

$$\left(\Delta x = x - a, \ \Delta f = f(x) - f(a) \right)$$

<u>Rm</u> $Q(x)$ is quadratic approximation of $f(x)$ near $(x-a)$

if (i) $Q(x)$ is a quadratic polynomial

(ii) $\lim\limits_{x \to 0} \dfrac{f(x) - Q(x)}{(x-a)^2} = 0$

pf of Chain Rule:

$$\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} =$$

$$= \lim_{u \to g(x_0)} \frac{f(u) - f(g(x_0))}{u - g(x_0)} \cdot \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

Let $\Delta x = x - x_0$, $\Delta u = g(x) - g(x_0)$
$= u - u_0$

$$\Delta y = f(g(x)) - f(g(x_0)) = f(u) - f(u_0)$$

Old proof: $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

let $\Delta x \to 0$

problem: devide by zero if $g(x) = g(x_0)$

(while $x \neq x_0$)

To fix the problem

we change $\lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u_0)$

to $\begin{cases} \Delta y = f'(u_0)\Delta u + \varepsilon_2 \Delta u \\ \lim\limits_{\Delta u \to 0} \varepsilon_2 = 0 \end{cases}$

Similarly, $\begin{cases} \Delta u = g'(x_0)\Delta x + \varepsilon_1 \Delta x \\ \lim\limits_{\Delta x \to 0} \varepsilon_1 = 0 \end{cases}$

$\therefore \Delta y = (f'(u_0) + \varepsilon_2)\Delta u$

$= (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\Delta x$

Note: $\Delta x \to 0 \Rightarrow \Delta u \to 0 \Rightarrow \lim\limits_{\Delta x \to 0} \varepsilon_2 = 0$

Let $\Delta x \to 0$

$$\Rightarrow \Delta y = f'(u_0) \cdot g'(x_0) \cdot \Delta x$$

$$+ (\underbrace{f'(u_0)\varepsilon_1 + g'(x_0)\varepsilon_2 + \varepsilon_1\varepsilon_2}_{= \varepsilon})\Delta x$$

$$= f'(u_0) g'(x_0) \Delta x + \varepsilon \Delta x$$

$$\lim_{\Delta x \to 0} \varepsilon = 0$$

$$\therefore \frac{dy}{dx} = f'(u_0) g'(x_0)$$

i.e. $\frac{d}{dx} f(g(x)) = f'(g(x_0)) \cdot g'(x_0)$