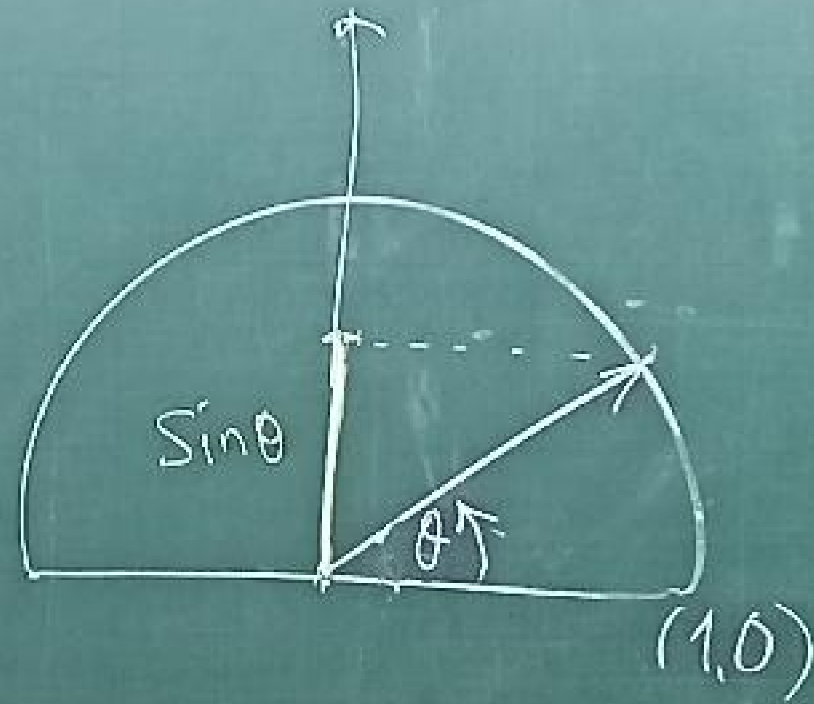
Inverse trigonometric functions

Definition of $\sin^{-1}x$:



$$\theta \in \mathbb{R} \xrightarrow{\sin\ (\text{NOT } 1\text{-}1)} x \in [-1,1]$$

Restrict $\downarrow$

$$\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \underset{\sin^{-1}}{\overset{\sin\ (1\text{-}1)}{\rightleftarrows}} x \in [-1,1]$$

$\sin(\sin^{-1}x) = x, \ \forall x \in [-1,1]$

$\sin^{-1}(\sin\theta) = \theta, \ \forall \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Eg1: $\sin^{-1}\left(\frac{1}{2}\right)$

$\sin\theta = \frac{1}{2}, \ \theta = ?$

$\theta = \underbrace{\frac{\pi}{6}}_{\sin^{-1}\left(\frac{1}{2}\right)}, \frac{5\pi}{6}, \frac{\pi}{6} \pm 2\pi, \dots$

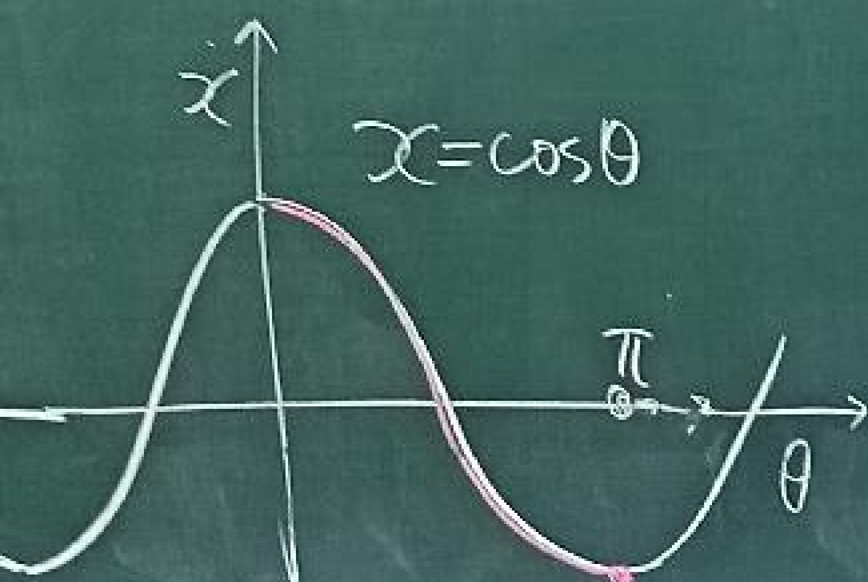
$\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right) \ \because \ \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

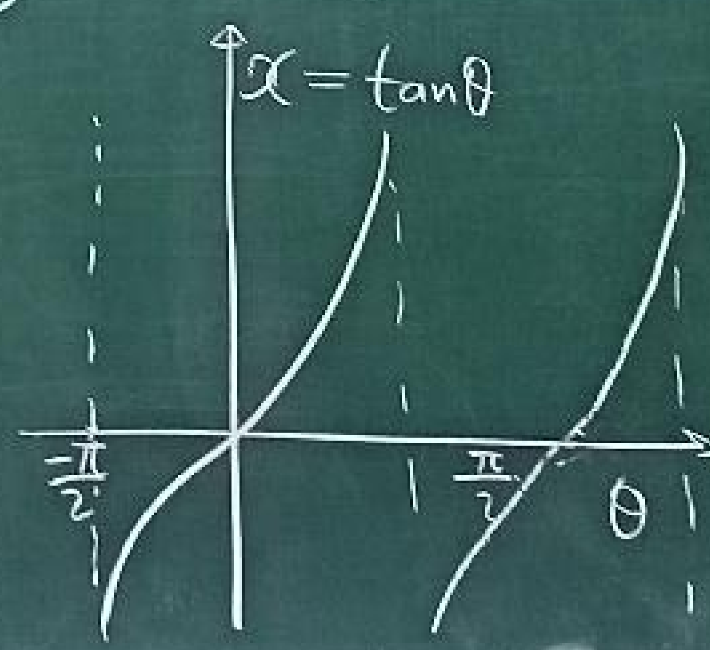
Eg2: $\underline{\sin^{-1}(\sin\pi)} \neq \pi$

$\begin{cases} \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \sin\theta = \sin\pi = 0 \end{cases} \Rightarrow \theta = 0$

$\sin^{-1}(\sin\pi) = \theta = 0$

f	$D_f = R_{f^{-1}}$	$\underset{f^{-1}}{\overset{f}{\rightleftarrows}}$	$R_f = D_{f^{-1}}$	f^{-1}
sin	$\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$		$x \in [-1, 1]$	$\sin^{-1}$
cos	$\theta \in [0, \pi]$		$x \in [-1, 1]$	$\cos^{-1}$
tan	$\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$		$x \in \mathbb{R}$	$\tan^{-1}$
cot	$\theta \in (0, \pi)$		$x \in \mathbb{R}$	$\cot^{-1}$
sec	$\theta \in [0, \pi] \setminus \{\frac{\pi}{2}\}$		$x \in (-\infty, -1] \cup [1, \infty)$	$\sec^{-1}$
csc	$\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$		$x \in (-\infty, -1] \cup [1, \infty)$	$\csc^{-1}$





Derivative of Trig. functions

(1) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\left.\frac{d}{d\theta}\sin\theta\right|_{x=\sin\theta}} = \frac{1}{\cos\theta}$

$(\theta = \sin^{-1}x)$

It remains to express $\cos\theta$ in terms of x.

$\begin{cases} \sin\theta = x, \\ \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}] \end{cases} \Rightarrow \cos\theta = ?$

$\sin^2\theta + \cos^2\theta = 1$, $\therefore \cos\theta = \pm\sqrt{1-x^2}$

Take "+" $\because \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$

(2) Similarly

$\frac{d}{dx}\cos^{-1}x = \frac{1}{-\sin\theta}$ $(\theta \in (0, \pi))$ $= \frac{-1}{\sqrt{1-x^2}}$ $x \in (-1, 1)$

(3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{\frac{d}{d\theta}\tan\theta} = \frac{1}{\sec^2\theta}$

$(x = \tan\theta)$

$= \frac{1}{1+x^2}, \quad x \in \mathbb{R}$

(4) $\frac{d}{dx}\cot^{-1}x$

(5) $\frac{d}{dx}\sec^{-1}x$: exercise.

(6) $\frac{d}{dx}\csc^{-1}x = \frac{1}{\frac{d}{d\theta}\csc\theta} = \frac{-1}{\csc\theta\cot\theta}$

$(\csc\theta = x)$

$\cot\theta = ?(x)$

$\cot^2\theta = \csc^2\theta - 1 = x^2 - 1$

$\cot\theta = \pm\sqrt{x^2-1}, \quad (|x| > 1)$

"$+$" or "$-$" ?

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \begin{matrix} >0 \\ <0 \end{matrix} \text{ if } \theta \in \frac{\mathrm{I}}{\mathrm{IV}}$$

$$\theta \in \frac{\mathrm{I}}{\mathrm{IV}} \Longleftrightarrow \begin{matrix} 0 < \theta < \frac{\pi}{2} \\ -\frac{\pi}{2} < \theta < 0 \end{matrix}$$

$$x = \csc\theta$$

$$\Longleftrightarrow \begin{matrix} x > 1 \\ x < -1 \end{matrix}$$

$$\therefore \cot\theta = \begin{cases} \sqrt{x^2-1}, & \text{if } x > 1 \\ -\sqrt{x^2-1}, & \text{if } x < -1 \end{cases}$$

$$\because \frac{d}{dx}\csc^{-1}\theta = \frac{-1}{x\left(\pm\sqrt{x^2-1}\right)}$$

$$= \frac{-1}{\begin{cases} +x\sqrt{x^2-1}, & (x>1) \\ -x\sqrt{x^2-1}, & (x<-1) \end{cases}} = \frac{-1}{|x|\sqrt{x^2-1}}$$