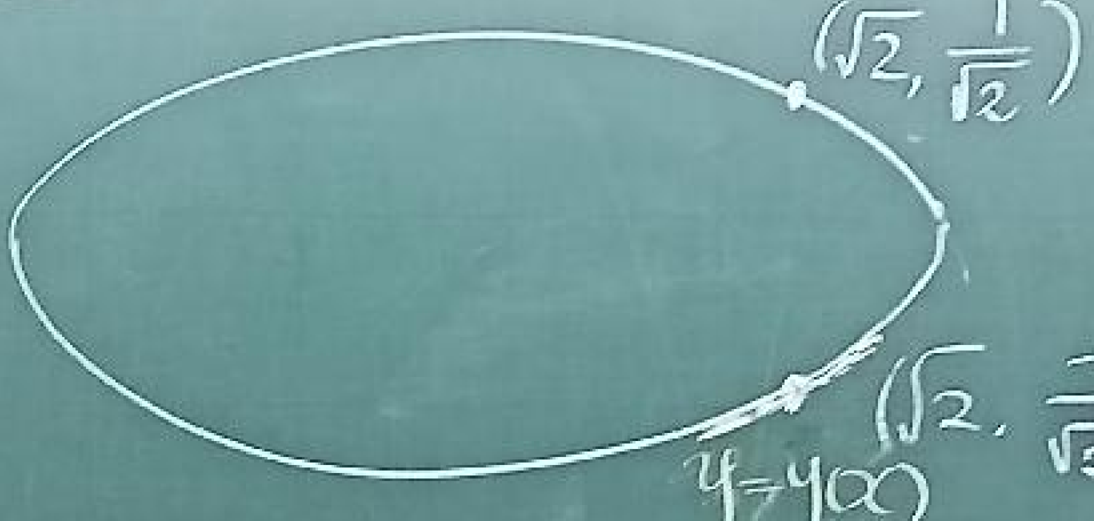
Implicit Differentiation

Goal: Find $\frac{dy}{dx}$ if the function $y(x)$ is not given explicitly, but implicitly by $F(x,y)=0$.

Eg1: Find the tangent and normal lines of

$$\frac{x^2}{4}+y^2=1 \text{ at } \left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$$

Sol



Method (a) (explicit diff.)

$$y^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow \quad y(x) = \pm\sqrt{1 - \frac{x^2}{4}}$$

(take "−")

$$\frac{d}{dx}\left(-\sqrt{1 - \frac{x^2}{4}}\right)\Bigg|_{x=\sqrt{2}} \qquad \left(\text{slope of tangent}\right)$$

$$= \frac{-1}{2}\left(1 - \frac{x^2}{4}\right)^{\frac{-1}{2}}\left(\frac{-x}{2}\right) = \frac{1}{2}$$

$\therefore$ tangent: $\dfrac{y-\frac{-1}{\sqrt{2}}}{x-\sqrt{2}} = \dfrac{1}{2}$

normal: $\dfrac{y-\frac{-1}{\sqrt{2}}}{x-\sqrt{2}} = \dfrac{-1}{\left(\frac{1}{2}\right)}$

Method (b): (Implicit diff.)

" $\dfrac{x^2}{4}+y^2=1$ implicitly defines $y(x)$ " means

$$\frac{x^2}{4}+y^2(x) \equiv 1$$

for all x near $\sqrt{2}$

$$\xrightarrow{\frac{d}{dx}} \frac{d}{dx}\left(\frac{x^2}{4}+y^2(x)\right) = \frac{d1}{dx} = 0$$

$$\Rightarrow \frac{x}{2} + 2y \cdot y' = 0 \quad (*)$$

$$y'\Big|_{(\sqrt{2}, \frac{-1}{\sqrt{2}})} = \frac{-x}{4y}\Big|_{(\sqrt{2}, \frac{-1}{\sqrt{2}})} = \frac{1}{2}$$

= slope of tangent

$\Rightarrow \begin{cases} \text{tangent line} \\ \text{normal line.} \end{cases}$

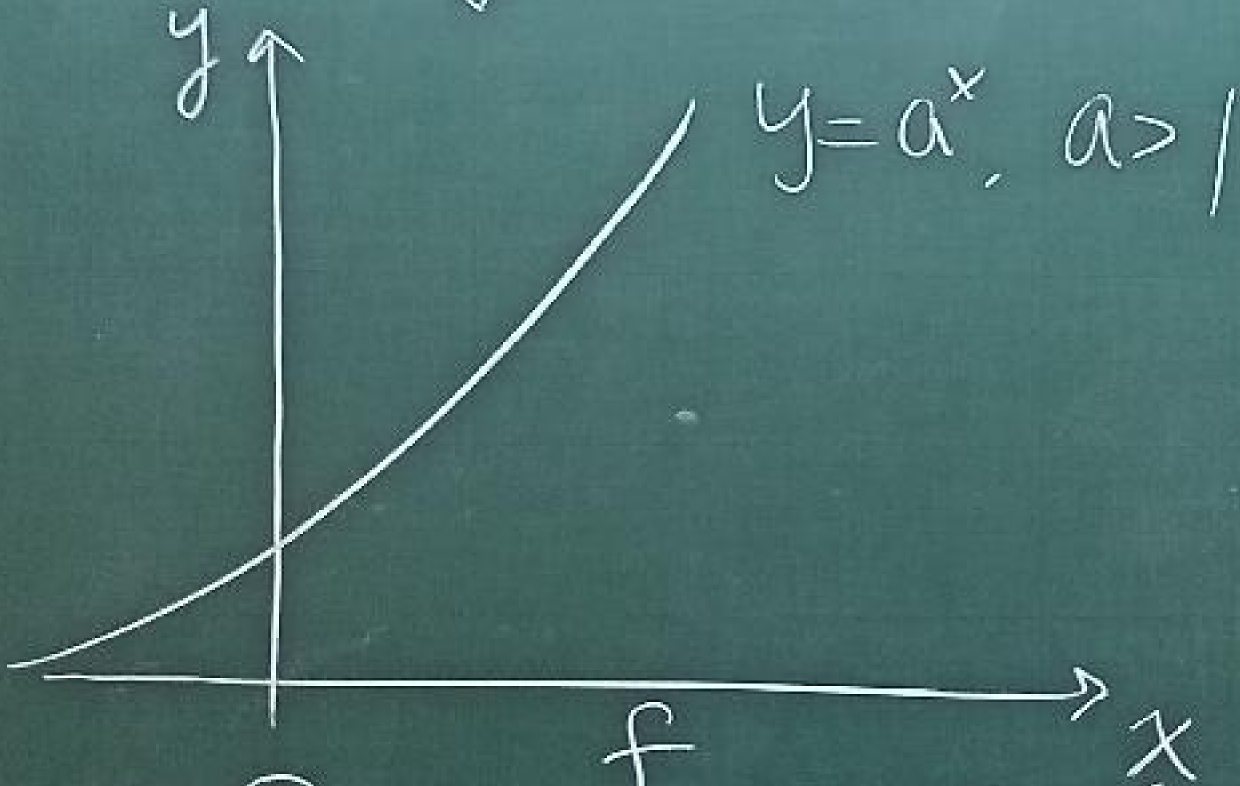
<u>Remark</u>: $y''\Big|_{(\sqrt{2}, \frac{-1}{\sqrt{2}})} = ?$

$$\xrightarrow{\frac{d}{dx}(*)} \frac{1}{2} + 2(y')^2 + 2y \cdot y'' = 0$$

Substitute $x = \sqrt{2}$, $y = \frac{-1}{\sqrt{2}}$

$$y' = \frac{1}{2} \Rightarrow y''_{(\sqrt{2}, \frac{-1}{\sqrt{2}})} = \frac{1}{\sqrt{2}}$$

Derivative of inverse functions and logarithm.



$$x \in \mathbb{R} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y = a^x > 0$$

$0 \quad x \quad \mathbb{R}$

$$x \overset{f}{\longmapsto} y$$
$$x \overset{f^{-1}}{\longleftarrow\!\shortmid} y$$

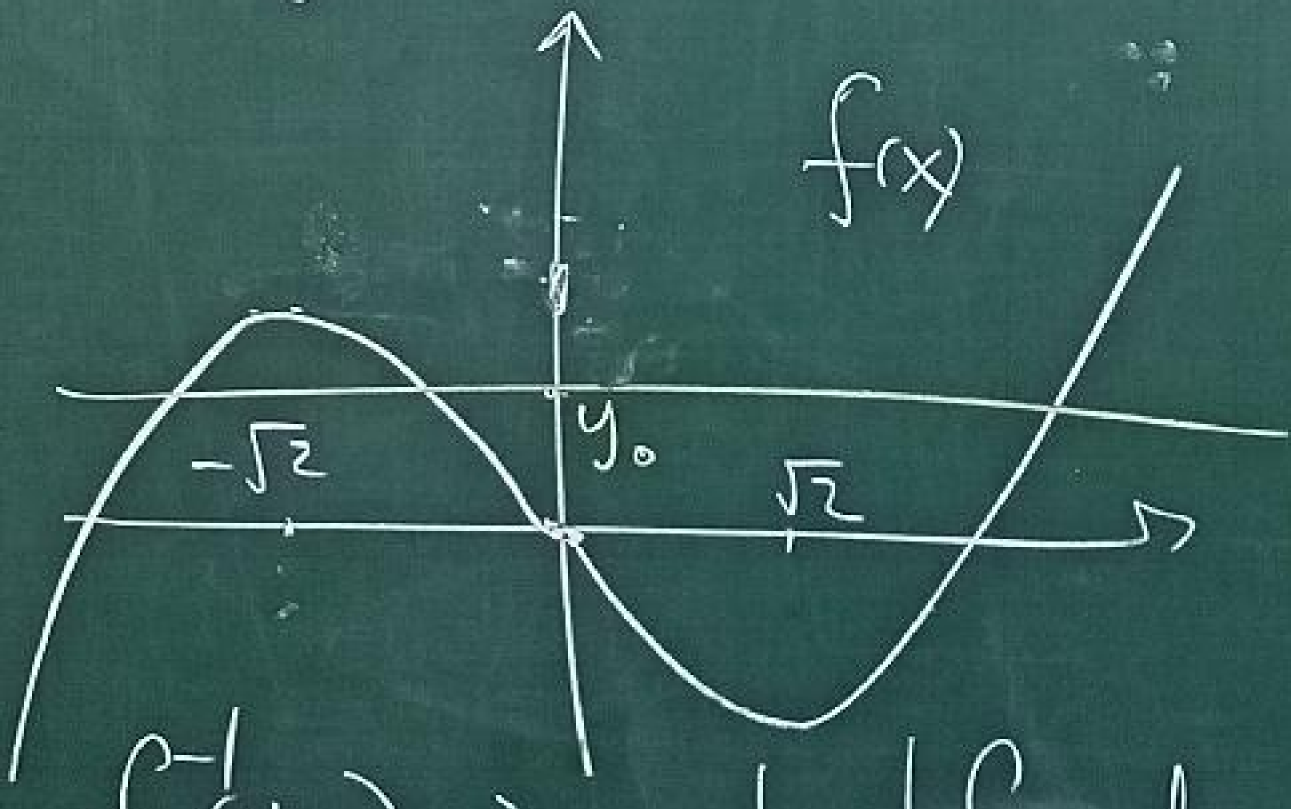
$0 \quad y \quad \mathbb{R}^+$

Remark: If $x_1 \neq x_2$

but $f(x_1) = f(x_2) = y$,

then we can NOT define $f^{-1}(y)$

Eg 1: $f(x) = x^3 - 3x^2 - 1$

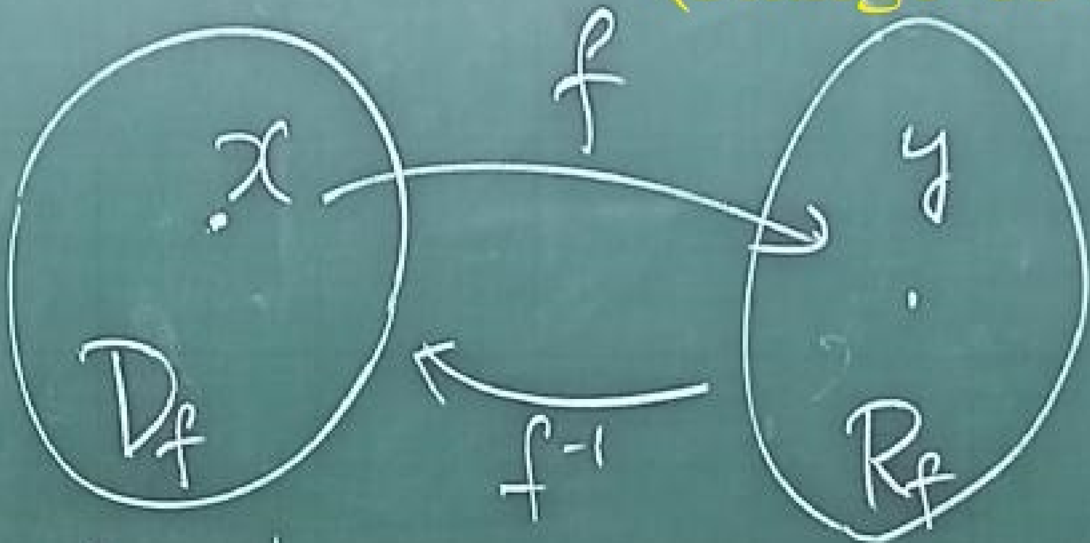


$f^{-1}(y_0)$ is not defined.

(Domain of f)

If $f: D_f \longmapsto R_f$

(Range of f)



Then "$f^{-1}: R_f \longmapsto D_f$ exists"

$$\Longleftrightarrow \begin{cases} f \text{ is one to one on } D_f \\ f \text{ maps } D_f \text{ \underline{onto} } R_f \end{cases}$$

One to one: $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

onto: $R_f = \{ f(x), x \in D_f \}$

Proposition $f: D_f \longrightarrow R_f$

If f^{-1} exists, then

(a): $f^{-1}(f(x)) = x$ for any $x \in D_f$

(b) $f(f^{-1}(y)) = y$ for any $y \in R_f$

Eg 2: The inverse function of $y = a^x$, $a>0$

is $x = \log_a y$ (textbook: $y = \log_a x$).

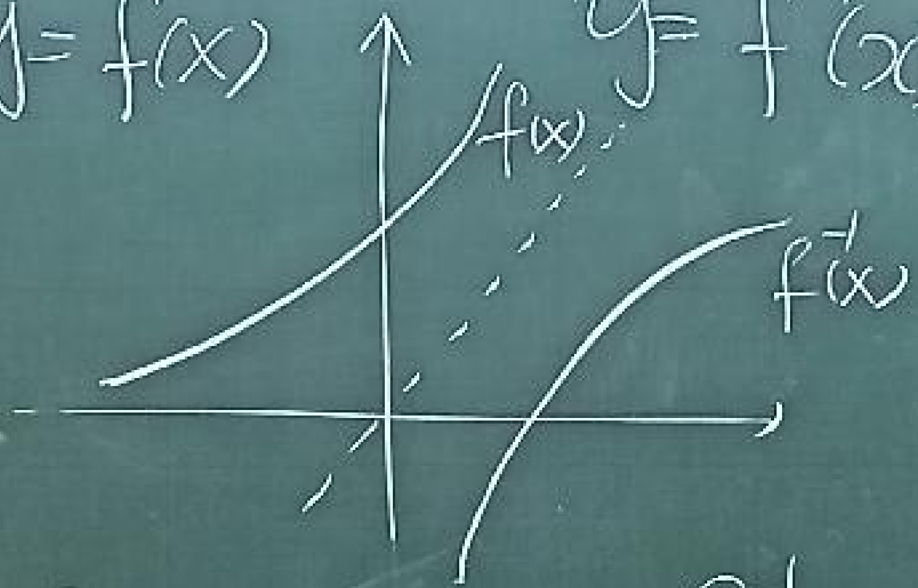
$\therefore$ $\log_a(a^x) = x$, $x \in \mathbb{R}$

$a^{\log_a y} = y$, $y \in \mathbb{R}^+$

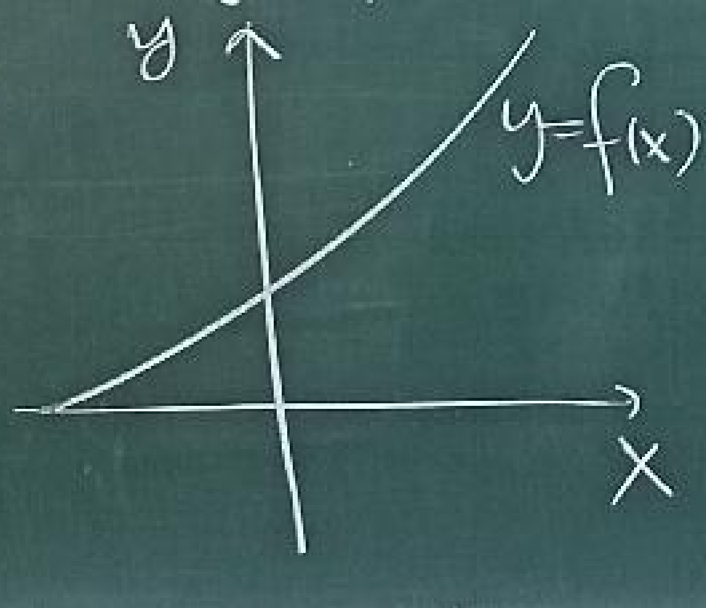
Notation difference

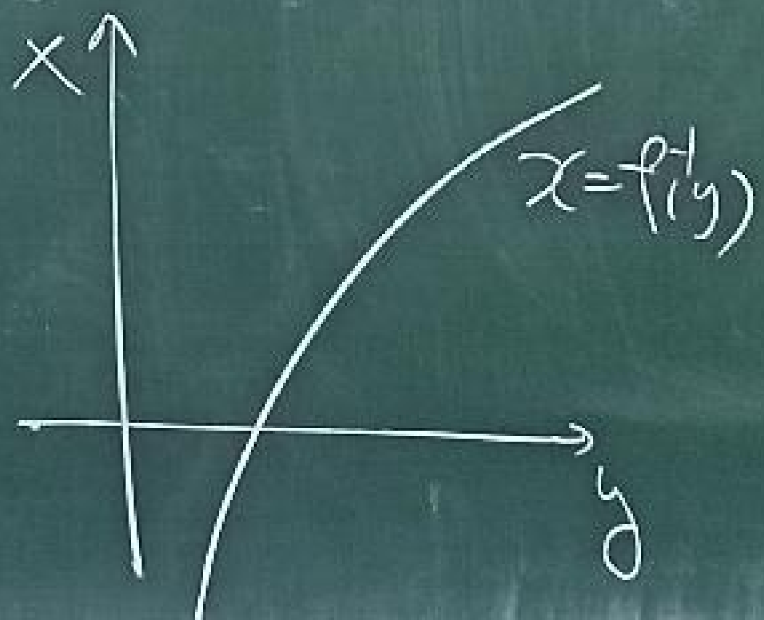
Original function, inverse function

textbook: $y=f(x)$, $y=f^{-1}(x)$



Here: $y=f(x)$, $x=f^{-1}(y)$





Derivative of f^{-1}.

$$f^{-1}(f(x)) = x \qquad x \in D_f$$

$$\frac{d}{dx} \Rightarrow \frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} \cdot f'(x) = 1$$

$$\therefore \frac{d}{dy} f^{-1}(y)\Big|_{y=f(x)} = \frac{1}{f'(x)}$$

$$\text{or } \frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)\big|_{x=f^{-1}(y)}}$$

Textbook:

$$\frac{d f^{-1}(x)}{dx}\Big|_{x=b} = \frac{1}{\frac{df(x)}{dx}\Big|_{x=f^{-1}(b)}}$$

Eg: The inverse function of $y = a^x$ is $x = \log_a y$

In particular, when $a = e$ (Euler number)

$$\log_e(\cdot) = \ln(\cdot)$$

(natural log)

$$\ln e^x = x, \quad x \in \mathbb{R}$$

$$e^{\ln y} = y, \quad y > 0$$

Note: the inverse function of $y = e^x$ is $y = \ln x$ (text book notation)

$$\frac{d}{dx}\ln x = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x}$$

(WRONG)

Current notation: $y = e^x$
$x = \ln y$

$$\underline{\underline{\frac{d}{dy}\ln y}} = \frac{1}{\frac{d}{dx}e^x \Big|_{x=\ln y}}$$

$$= \frac{1}{e^x \big|_{x=\ln y}} = \frac{1}{e^{\ln y}} = \boxed{\frac{1}{y}}$$ correct