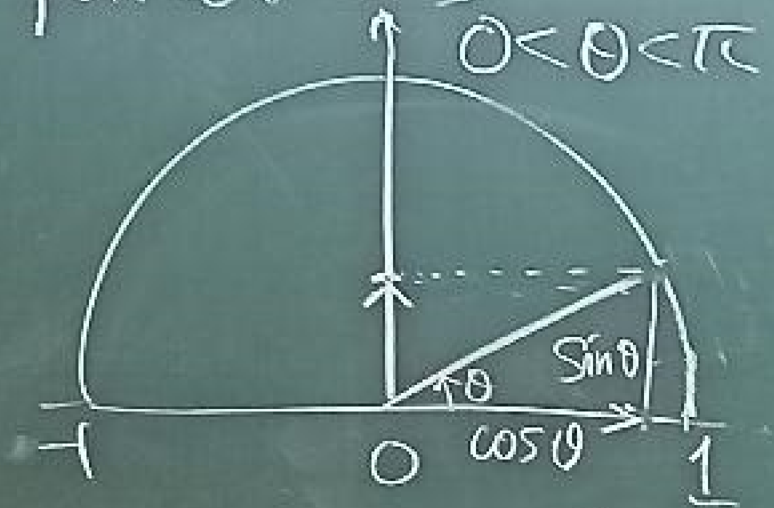
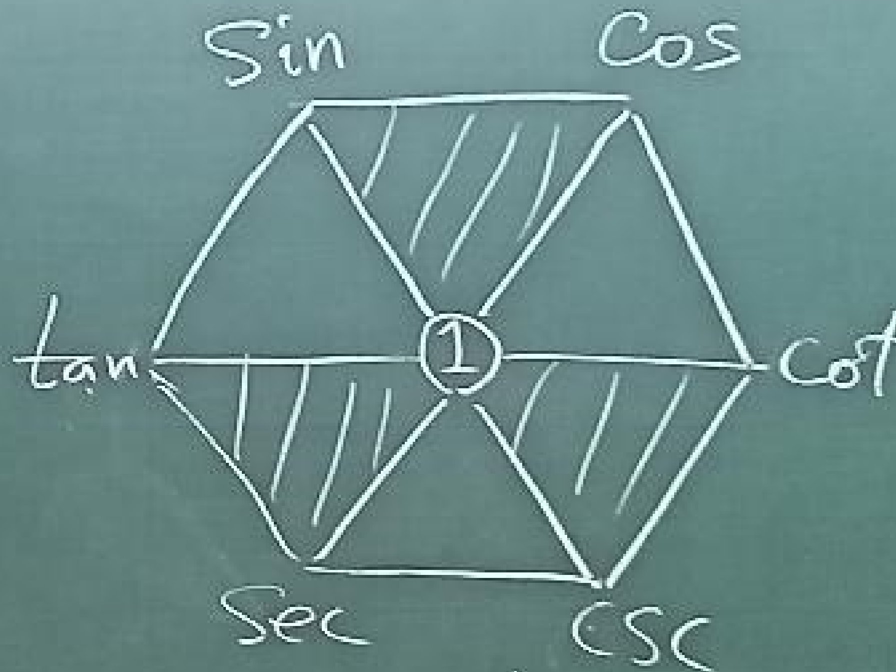
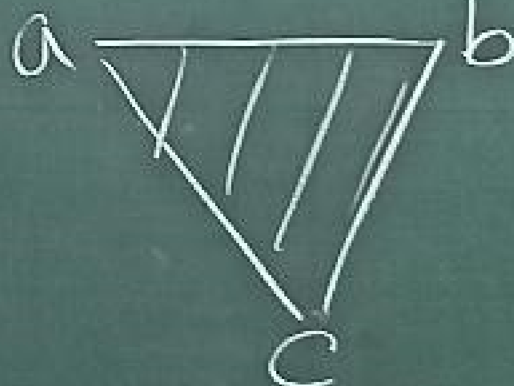
Derivatives of Trigonometric functions





$\sin\theta$ = projection on y axis

($\cos\theta$) (x)



$\tan\theta = \sin\theta / \cos\theta$

$\cot\theta = \cos\theta / \sin\theta$

$\sec\theta = 1 / \cos\theta$

$\csc\theta = 1 / \sin\theta$

$$a^2 + b^2 = c^2$$

When $-\pi < \theta < 0$

$\sin\theta = -\sin(-\theta)$, $\cos\theta = \cos(-\theta)$

We also have

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Eg1: $\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

$$\sin(\theta \pm \pi) = -\sin\theta$$

$$\cos(\theta \pm \pi) = -\cos\theta$$

Eg2: $f(x) = \sin x$, $f'(x) = ?$

Sol: $f'(x) = \lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\,\frac{\sin h}{h}\right)$$

$$= \sin x \underbrace{\lim_{h\to 0} \cancel{\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}}}_{0} + \cos x \lim_{h\to 0}\frac{\sin h}{h} = \cos x$$

Similarly, we have $\frac{d}{dx}\cos x = -\sin x$

Therefore (Memorize)

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \cdots\cdots = -\csc^2 x$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cos x\, 1' - 1(\cos x)'}{\cos^2 x} = \tan x \sec x$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \cdots\cdots = -\cot x \csc x$$

Eg 3. $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = ?$

Ans $= \dfrac{x(\sin x)' - (\sin x)x'}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

(Exercise. $\lim_{x\to 0} \frac{d}{dx}\left(\frac{\sin x}{x}\right) = ?$

Chain Rule: $\frac{d}{dx} f(g(x)) = ?$

Thm **If** $y = f(u)$ is differentiable at $u = g(x)$ **and** $g(x)$ is diff. at x, **then** $y = f \circ g(x) = f(g(x))$ is differentiable at x

and $\frac{dy}{dx} = \frac{dy}{du}\Big|_{u=g(x)} \cdot \frac{du}{dx}$

$\left(\text{i.e. } \frac{d}{dx} f(g(x)) = \frac{df(u)}{du}\Big|_{u=g(x)} \cdot \frac{d}{dx} g(x) \right)$

$\underline{\text{Eg4}}$ $\frac{d}{dx}\sin(\tan x)$

$= \frac{d}{du}\sin u\Big|_{u=\tan x} \cdot \frac{d}{dx}\tan x$

$= \cos(\tan x) \cdot \sec^2 x$

$\underline{\text{Eg5.}}$ $(\cot^7 x)' = ?$

$\underline{\text{Ans}} = \frac{du^7}{du}\Big|_{u=\cot x} \cdot (\cot x)'$

$= 7\cot^6 x \cdot (-\csc^2 x)$

$\left(\begin{array}{l} \text{i.e. } f(u) = u^7,\ g(x) = \cot x \\ \cot^7 x = f(g(x)) \end{array}\right)$

Eg6 $\frac{d}{dx}\sin(x^2+e^x)$

$= \cos(x^2+e^x)\cdot(x^2+e^x)'$

$= (2x+e^x)\cdot\cos(x^2+e^x)$

Eg7 $\frac{d}{dx}\sin\left(\cos(x^2+1)\right)$

$= \cos\left(\cos(x^2+1)\right)\cdot\left(\cos(x^2+1)\right)'$

$= \cos\left(\cos(x^2+1)\right)\cdot\left(-\sin(x^2+1)\cdot(x^2+1)'\right)$

$= -\cos\left(\cos(x^2+1)\right)\cdot\sin(x^2+1)\cdot 2x$

Remark:

* $\frac{d}{dx}(u(x))^n = n u^{n-1}(x) \cdot u'(x)$

* $\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$

* $\frac{d}{dx} \sin(u(x)) = \cos(u(x)) \cdot u'(x)$

etc.

Eg 7. $\frac{d}{dx}\left(\frac{1}{v(x)}\right) = \frac{d}{dx}\left((v(x))^{-1}\right)$

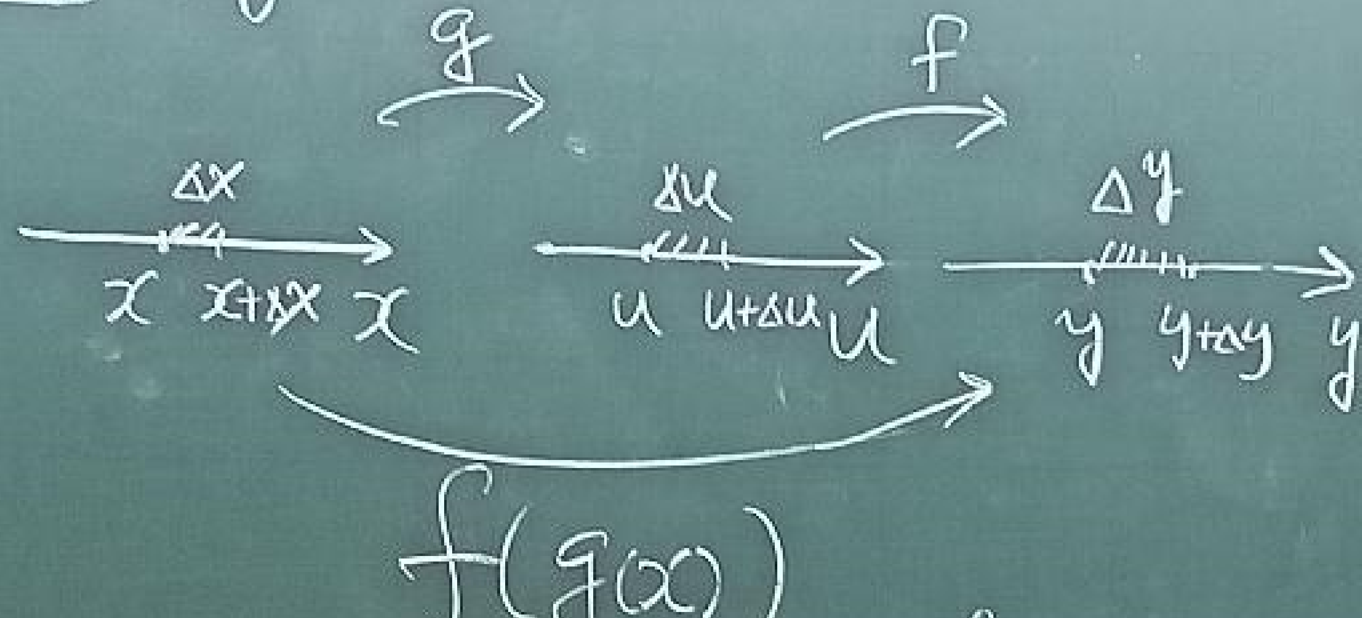
$= -(v(x))^{-2} v'(x) = \frac{-v'(x)}{v^2(x)}$

Eg 8: $\frac{d}{dx}\sqrt{1+x^2} = \frac{d}{dx}(1+x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$

Eg 9: $\frac{d}{dx} e^{x^e} = e^{x^e} \cdot (x^e)' = e^{x^e} \cdot e x^{e-1}$

pf (of Chain Rule)



$$g'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$f'(u) = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{f(\underline{u+\Delta u}) - f(u)}{\Delta u}$$

Note: $\Delta u = g(x+\Delta x) - g(x)$

$\therefore \underline{g(x+\Delta x)} = g(x) + \Delta u = \underline{u + \Delta u}$

$$\frac{d}{dx} f(g(x)) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u}\right) \cdot \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x}\right)$$

Since g is diff. at x

$\Rightarrow$ g is cont. at x

i.e. " $\Delta x \to 0 \Rightarrow \Delta u \to 0$ "

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$$

$$\therefore f(g(x))' = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)$$

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= f'(u)\Big|_{u=g(x)} \cdot g'(x)$$

$$\therefore f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{du} f(u) \Big|_{u=g(x)} \cdot \frac{d}{dx} g(x)$$

<u>Remark</u>: $\Delta u = g(x+\Delta x) - g(x)$

it can happen that

$\Delta x \neq 0$, but $\Delta u = 0$,

this is a ~~mior~~ minor problem in the proof. See section 3.11 for a correct and complete proof.