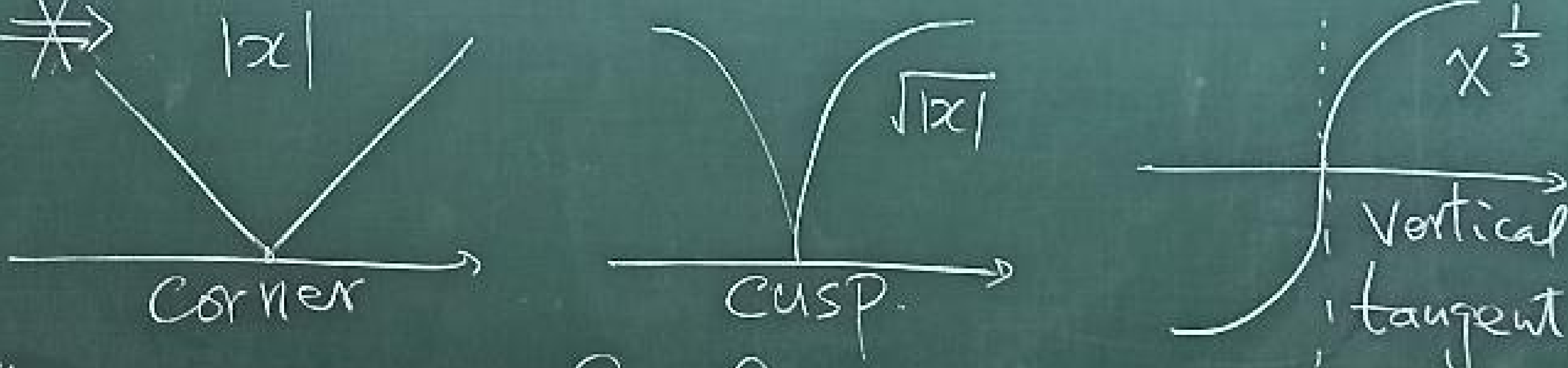
Relation between Continuity and differentiability.

$f(x)$ is cont. at x_0 $\overset{\not\Rightarrow}{\Leftarrow}$ $f(x)$ is diff. at x_0

"$\not\Rightarrow$"



"$\Leftarrow$" <u>Thm</u> If f has a derivative at c then f is continuous at c.

pf

f is differentiable at c

$$\Longleftrightarrow \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \text{ exists } (=f'(c)) \quad (1)$$

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f is continuous at c

$$\Longleftrightarrow \lim_{x\to c}\left(f(x)-f(c)\right)=0 \quad (2)$$

$$(1) \Rightarrow \lim_{x\to c} f(x)-f(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c)$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0$$

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Differentiation Rules

Eg1 $f(x) = 1$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$

Eg2. $f(x) = x^3$. $f'(x) = ?$

Ans: $f'(x) = \lim_{h \to 0} \dfrac{(x+h)^3 - x^3}{h}$.

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= 3x^2$$

In general if $f(x)=x^n$, $n\in\mathbb{N}$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^n}+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\cdots-\cancel{x^n}}{h}$$

$$=nx^{n-1}(+0+0+\cdots)$$

Eg 3: $f(x) = x^{\frac{1}{3}}, \; x \neq 0$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}$$

$$(a-b)(a^2+ab+b^2) = a^3 - b^3$$

$$= \lim_{h \to 0} \frac{\left((x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}$$

$$= \lim_{h \to 0} \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}$$

$$= \frac{1}{3} \frac{1}{x^{\frac{2}{3}}} = \frac{1}{3} x^{\frac{-2}{3}} = \frac{1}{3} x^{\frac{1}{3}-1}$$

Remark: In fact

$\frac{d}{dx} x^n = n x^{n-1}$ holds for

$n = \frac{q}{p}$, $p, q \in \cancel{N}$ $\mathbb{Z}$ (integers), $p \neq 0$

and also for $n \in \mathbb{R}$ (later)

Derivative Rules

$$\frac{d}{dx}(C\, u(x)) = C \frac{d}{dx} u(x) \quad \cdots (1)$$

C = constant

$$\frac{d}{dx}(u(x) \pm v(x)) = \frac{du}{dx} \pm \frac{dv}{dx} \quad \cdots (2)$$

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'v + uv' \quad \cdots (3)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \quad \cdots (4)$$

$(v(x) \neq 0)$

Pf of (3):

$$(uv)' = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} u(x+h)\frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} v(x)$$

$$= u(x)v'(x) + u'(x)v(x)$$

Pf of (4):

$$\frac{u}{v} = u \cdot \left(\frac{1}{v}\right), \quad v(x) \neq 0$$

From (3), it remains to evaluate

$$\left(\frac{1}{v}\right)'(x) = \lim_{h \to 0} \frac{\frac{1}{v(x+h)} - \frac{1}{v(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{v(x) - v(x+h)}{v(x+h)\,v(x)}}{h}$$

$$= -\lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \cdot \frac{1}{v(x+h)\,v(x)}$$

$$= -\frac{v'}{v^2}(x) \Longrightarrow (4)$$

Eg4: $f(x) = x^4 - 2x^2 + 2$

Find all horizontal tangents of $y = f(x)$.

Ans: Solve for x from $f'(x) = 0$

$$f'(x) = 4x^3 - 4x + 0$$

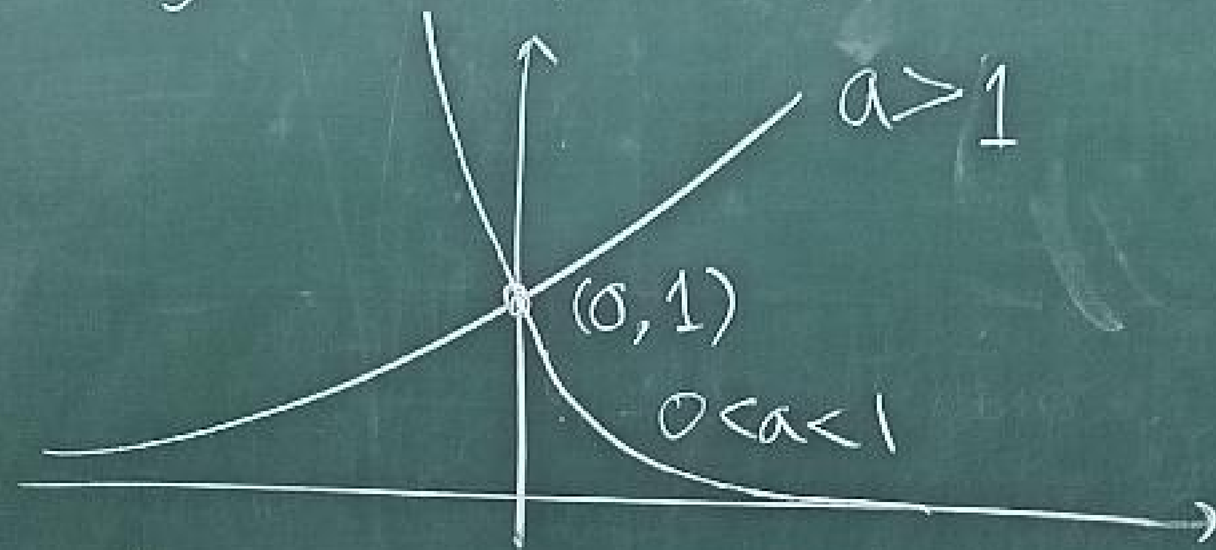
$$f'(x) = 0 \Leftrightarrow x = 0, \pm 1$$

horizontal tangents at $(0, f(0))$ $(1, f(1))$ and $(-1, f(-1))$ are

$$\frac{y-2}{x-0} = 0, \quad \frac{y-1}{x-1} = 0 \text{ and } \frac{y-1}{x+1} = 0$$

Exponential functions

$$y = a^x, \quad x \in \mathbb{R}, \quad a > 0$$



$$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \Big(\underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{g(a)} \Big)$$

$$= g(a)\, a^x$$

Here $g(a) = \lim_{h\to 0} \frac{a^h - 1}{h}$

(if the limit exists)

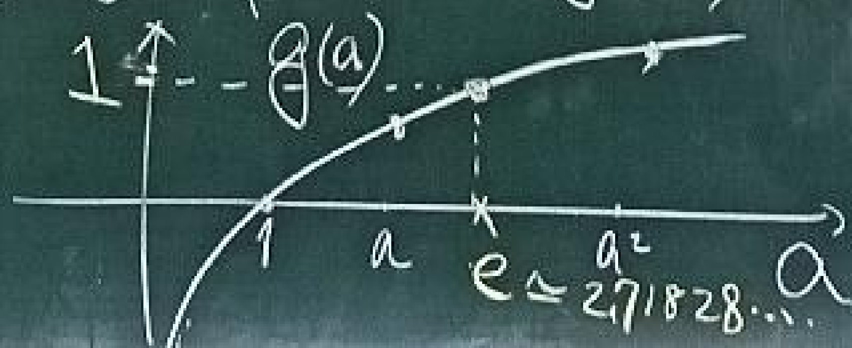
$$g(1) = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$g(a^2) = \lim_{h\to 0} \frac{a^{2h} - 1}{h}$$

$$= \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right)(a^h + 1) = 2g(a)$$

$a > 1 \Rightarrow g(a) \geq 0$ $\left(\begin{matrix}\text{In fact} \\ > 0\end{matrix}\right)$

$0 < a < 1 \Rightarrow g(a) < 0$



Define the Euler number "e"

satisfying $g(e) = 1$

i.e. $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, $e \simeq 2.71828\ldots$

or $\frac{d}{dx} e^x = e^x$

<u>Eg 5</u>: $f(x) = e^{-x}$, $f'(x) = ?$

<u>Ans</u>: $\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{e^x}\right)$

from (4)

$\Rightarrow f'(x) = \frac{e^x \cdot 1' - 1(e^x)'}{(e^x)^2}$

$= -e^{-x}$