

Recall: $\lim_{x \rightarrow c} f(x) = L$

For every $\epsilon > 0$, there exists
a corresponding $\delta > 0$ such that
" $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$ "
(x is sufficiently close to c)

Def: $\lim_{x \rightarrow \pm\infty} f(x) = L$

For every $\epsilon > 0$, there exists
a corresponding $M \in \mathbb{R}$, such that
" $x > M \Rightarrow |f(x) - L| < \epsilon$ "
" $x < N \Rightarrow |f(x) - L| < \epsilon$ "
(x is sufficiently close to $\pm\infty$)

Def: $\lim_{x \rightarrow c, c^+, c^-} f(x) = \begin{matrix} +\infty \\ (-\infty) \end{matrix}$

($|f(x) - L| < \varepsilon$ means " $f(x)$ is close to L ")

Replace it by " $f(x)$ is close to $\begin{matrix} +\infty \\ (-\infty) \end{matrix}$ ")

For every $\underline{B > 0}$ ⁽¹⁾ there exists a corresponding $\delta > 0$, such that

" $0 < |x - c| < \delta \Rightarrow f(x) > B$ "
 $0 < x - c < \delta$ $(\leftarrow -B)$ (2)
 $-\delta < x - c < 0$

(or (1) $\rightarrow B \in \mathbb{R}$, (2) $\rightarrow \begin{matrix} > B \\ < B, < -B \end{matrix}$)

$$\text{Eg 1 } \lim_{x \rightarrow 0^+} 2^{\frac{1}{x}} = +\infty$$

$$\text{Eg 2 } \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

$$\text{Eg 3 } \lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

does not exist. (oscillatory)

$$\begin{aligned} \text{Eg 4 } \lim_{\theta \rightarrow \frac{\pi}{2}^+} \tan \theta &= \lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{\sin \theta}{\cos \theta} \\ &= -\infty \end{aligned}$$

Eg 4. Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

If For every $\varepsilon > 0$

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \frac{1}{x} < \varepsilon$$

$$\Leftrightarrow \left\{ \begin{array}{l} 0 < \frac{1}{x} < \varepsilon \\ \text{or} -\varepsilon < \frac{1}{x} < 0 \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} x > \frac{1}{\varepsilon} \\ \text{or} x < -\frac{1}{\varepsilon} \end{array} \right.$$

$$\Leftarrow x > M$$

(take $M = \frac{1}{\varepsilon}$ will do)

The most important sentence in the proof

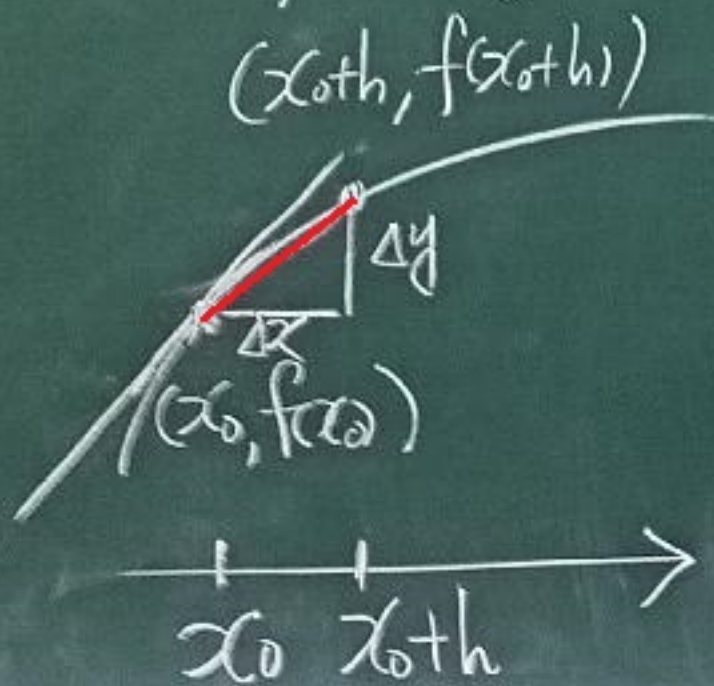
Chap 3.

Def. Derivative of f at x_0

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(= \lim (slope of secant (割線))) = $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

= slope of tangent (切線)



$y = f(x)$
When $x = \text{time}$
 $f'(x_0) = \text{rate of change of } y \text{ at } x_0$

$f'(x_0)$ = derivative of f at x_0

$f'(x)$ = derivative of f as
a function of x

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

Notation: $\frac{d}{dx} f(x)$, $\frac{df}{dx}$

$\frac{dy}{dx}$, $D_x f$, $f'(x)$

$$\frac{d}{dx} f(x_0) = \left. \frac{d}{dx} f(x) \right|_{x=x_0} = \left. \frac{dy}{dx} \right|_{x=x_0}$$

$$\text{Eg 5. } f(x) = \frac{1}{x}, \quad x \neq 0$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{(x+h)x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

$$\text{Ex 6: } f(x) = \sqrt{x}, \quad x \geq 0$$

$$f'(x) = ?$$

Ans if $x > 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} + \sqrt{x})(\sqrt{x+h} - \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\text{If } x=0 \quad \lim_{h \rightarrow 0^+} \frac{\sqrt{h+0} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = +\infty$$