Continuity

Suppose $f(x)$ is defined on $[a, b]$

If $c \in (a, b)$, then f is

$(\underline{c \in (a, b]}, \underline{c \in [a, b)})$

continuous at c, if

(<u>left continuous</u>, <u>right continuous</u>)

$$\lim_{x \to c} f(x) = f(c)$$

$(\underline{x \to c^-}, \underline{x \to c^+})$

That is, for any $\varepsilon > 0$,

there exists a corresponding $\delta > 0$

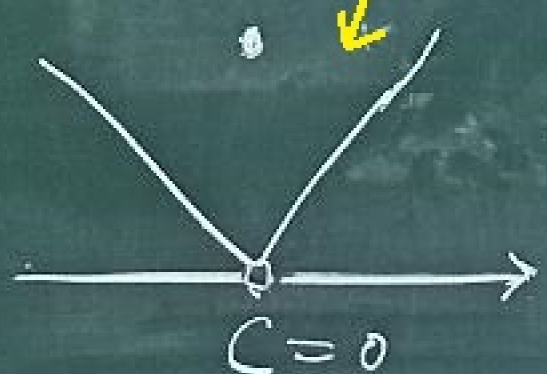
such that

$$\text{“} |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \text{”}$$

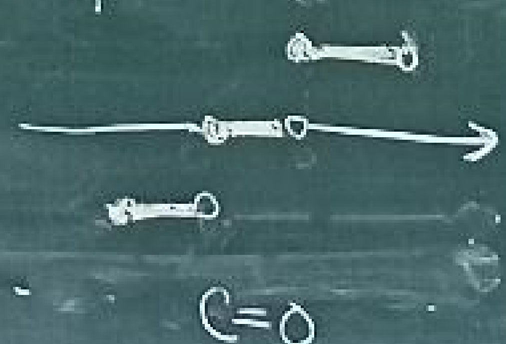
$$(\underline{c - \delta < x \le c}, \ \underline{c \le x < c + \delta})$$

(continuous at c)

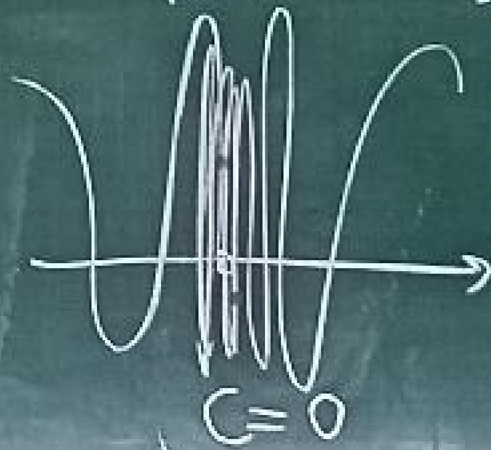
(Not continuous at c)



$f(x) = \lfloor x \rfloor$



$f(x) = \sin\frac{1}{x}$



<u>Remark</u>: f is a continuous function if f is continuous at every point of its domain.

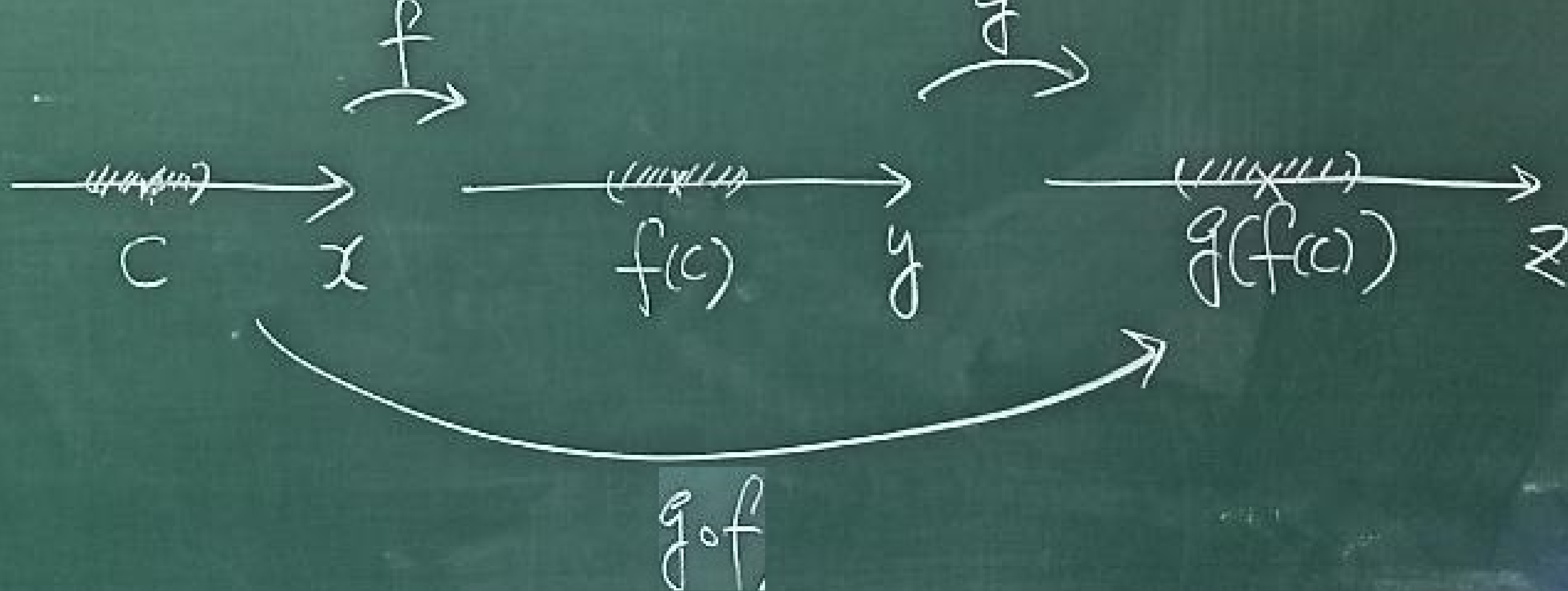
Basic properties of continuous functions (See Section 2.5, Thm 8)

<u>Remark</u>: polynomials, rational functions, trigonometric functions are all continuous at the interior of their domain of definition,

Thm [Composite of continuous functions]

If f is continuous at c, g is cont. at $f(c)$. then $(g\circ f)(x) = g(f(x))$ is continuous at c.

i.e. $\displaystyle\lim_{x\to c} g(f(x)) = g(f(c))$



Pf: To show that $g \circ f$ is continuous, we need to show for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

"$|x-c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"

To do this, we note that, for any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that,

$(*_1)$ "$|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$"

(g is continuous at $f(c)$)

For this $\delta_1 > 0$, there exists a corresponding $\delta > 0$ such that

$(*_2)$ "$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$"

(f is continuous at c)

$(*_1) + (*_2) \Longrightarrow$

"$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$

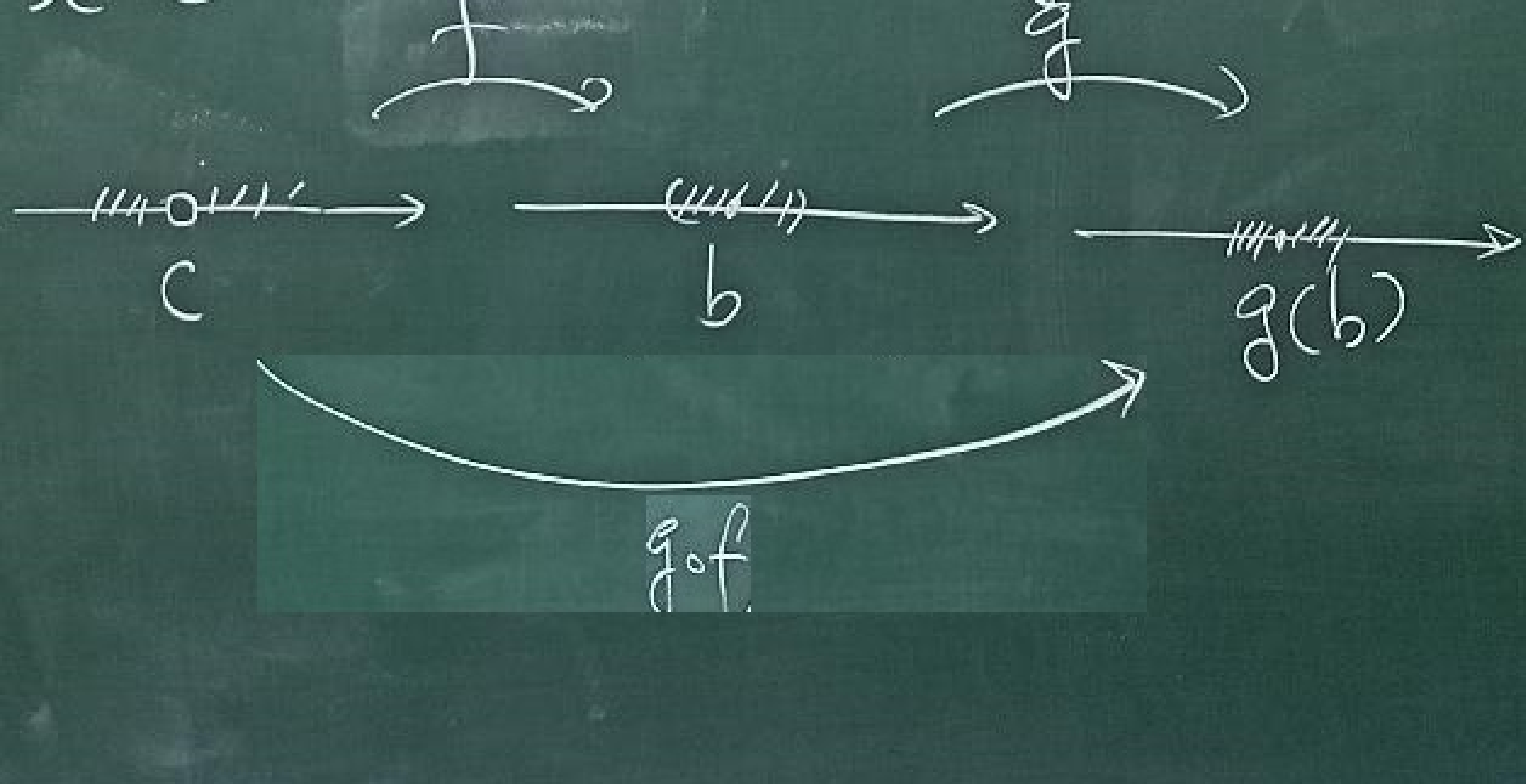
$\Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"

Eg1: $\lim\limits_{x \to 0} \sqrt{x+1} \cdot 2^{\sin x} = \sqrt{0+1} \cdot 2^{\sin 0} = 1$

Since both $\sqrt{x+1}$ and $2^{\sin x}$ are continuous function (Thm 9), so is $\sqrt{x+1} \cdot 2^{\sin x}$ (Thm 8)

__Remark__ Similarly, if g is continuous at b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) \left(= g\left(\lim_{x \to c} f(x)\right) \right)$$



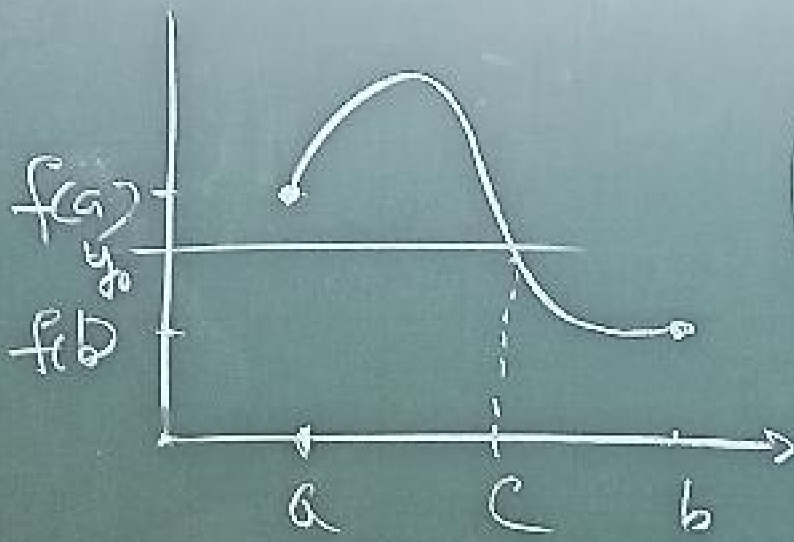
Thm [Intermediate Value Thm]

f is continuous on $[a, b]$

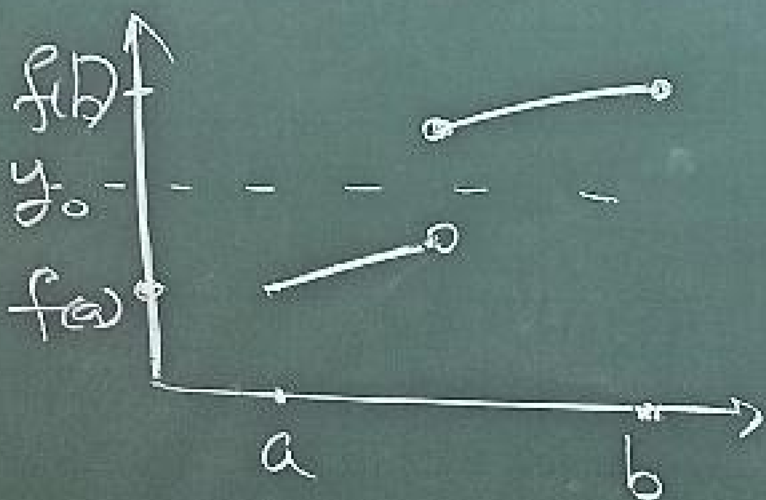
$\Longrightarrow$ f takes any value between $f(a)$ and $f(b)$.

i.e. if y_0 is any value between $f(a)$ and $f(b)$, then there exists a $c \in [a,b]$ such that

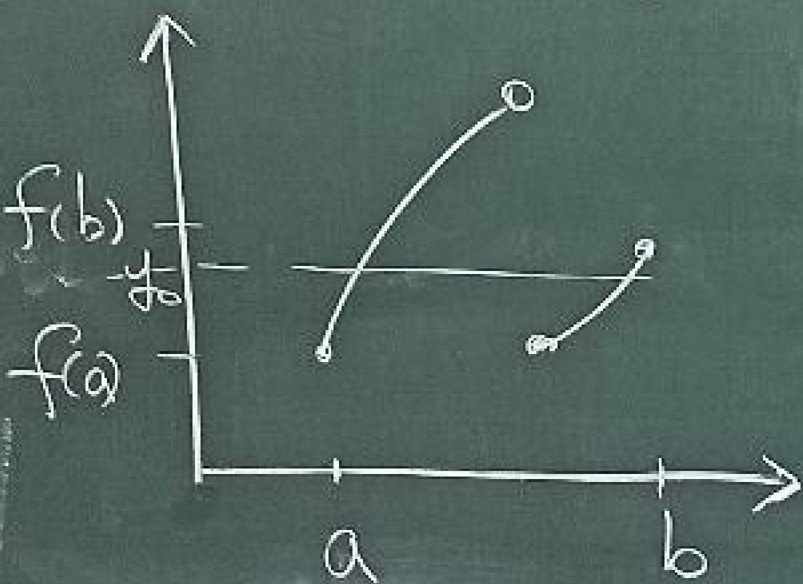
$$f(c) = y_0$$



(Typical case)



(I.V.T. may fail if f is not cont. on [a,b])



(f takes any value between f(a) and f(b) $\not\Rightarrow$ f is cont. on [a,b])

Eg 2. Show that $f(x) = x^3 - x - 1 = 0$ has a root in $(1, 2)$

Sol. $f(1) = -1$, $f(2) = 5$.

$\xrightarrow{\text{I.V.T.}} \exists x \in (1, 2)$. $f(x) = 0$.

(存在)

Eg 3. Show that

$\sqrt{2x+5} = 4 - x^2$ has a solution.

Sol. Let $f(x) = \sqrt{2x+5} - (4 - x^2)$

$f(0) = \sqrt{5} - 4 < 0$, $f(2) = 3 - 0 > 0$

$\text{I.V.T.} \Rightarrow \exists x \in (0, 2)$. $f(x) = 0$. $\left(\sqrt{2x+5} = 4 - x^2\right)$