Eg1 $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

Prove that $\lim\limits_{x \to 2} = 4$

~~pf~~ For any $\varepsilon > 0$, $(0 < \varepsilon \leq 4)$

we need ~~Need~~ to find $\delta > 0$

such that

$$|x^2 - 4| < \varepsilon, \quad \begin{pmatrix} \text{when} \\ x \neq 2 \end{pmatrix}$$

$$\Leftarrow \; 0 < |x - 2| < \delta$$

Only the "<=" part of "<=>" is relevant.

$$|x^2-4|<\varepsilon, (x\neq 2)$$

$$\Leftrightarrow -\varepsilon < x^2-4 < \varepsilon, (x\neq 2)$$

$$\Leftrightarrow 4-\varepsilon < x^2 < 4+\varepsilon, (x\neq 2)$$

$$\Leftrightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}, (x\neq 2)$$

(Need $4-\varepsilon \geq 0$)

or $\underline{-\sqrt{4+\varepsilon} < x < -\sqrt{4-\varepsilon}}$

Not the interval we want

$$\Leftarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}, \ x\neq 2$$

$$\Leftrightarrow \underbrace{\sqrt{4-\varepsilon}-2}_{\text{"}-\text{"}} < x-2 < \underbrace{\sqrt{4+\varepsilon}-2}_{\text{"}+\text{"}} \ (x\neq 2)$$

Find $\delta>0$ such that

$\Leftarrow 0<|x-2|<\delta$

We need

(i) $\sqrt{4-\varepsilon}-2<0$ (satisfied if $\because 0<\varepsilon\leq 4$)

(ii) $(-\delta,\delta)\subset\left(\sqrt{4-\varepsilon}-2,\sqrt{4+\varepsilon}-2\right)$

$\therefore$ Take $\delta=\min\left(2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\right)$

Conclusion: if $0<\varepsilon\leq 4$, δ (the δ above)

then $0<|x-2|<\delta \Rightarrow |f(x)-4|<\varepsilon$

For $\varepsilon>4$, take the δ obtained from $\varepsilon=3$, then

$0<|x-2|<\delta \Rightarrow |f(x)-4|<3<4$

Eg 2 If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$

Prove that $\lim_{x\to c}(2f(x) + g(x)) = 2L + M$

Pf: For any $\varepsilon > 0$, there exists $\delta_1 > 0$, $\delta_2 > 0$ such that

$$0 < |x-c| < \delta_1 \Rightarrow \frac{-\varepsilon}{3} < f(x) - L < \frac{\varepsilon}{3}$$

$$0 < |x-c| < \delta_2 \Rightarrow \frac{-\varepsilon}{3} < g(x) - M < \frac{\varepsilon}{3}$$

take $\delta = \min(\delta_1, \delta_2)$, then

$$0 < |x-c| < \delta \Rightarrow \begin{cases} -2\varepsilon/3 < 2(f(x) - L) < 2\varepsilon/3 \\ -\varepsilon/3 < g(x) - M < \varepsilon/3 \end{cases}$$

$$\Rightarrow -\varepsilon < 2f(x) + g(x) - (2L+M) < \varepsilon \Rightarrow |2f(x) + g(x) - (2L+M)| < \varepsilon$$

One sided limit

Suppose $f(x)$ is defined

on $(c, c+a)$, $a>0$

$\left(\underline{(c-a, c)} \right)$

Def: $\lim\limits_{x \to c^{\underline{\pm}}} f(x) = L$

if for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

"$c < x < c+\delta \Longrightarrow |f(x) - L| < \varepsilon$"

$\left(\underline{c - \delta < x < c} \right)$

Eg 3. Show that $\lim_{x \to 0^+} \sin\frac{1}{x}$ does not exist.

Pf Let $a_n = \frac{1}{n\pi}$

$b_n = \frac{1}{(2n+\frac{1}{2})\pi}$, $c_n = \frac{1}{(2n-\frac{1}{2})\pi}$

$n = 1, 2, 3, \cdots$

$\Rightarrow \sin\frac{1}{a_n} = 0$, $\sin\frac{1}{b_n} = 1$, $\sin\frac{1}{c_n} = -1$

for any $\delta > 0$, there are infinitly infinitely many a_n, b_n and c_n on $(0, \delta)$

$\Rightarrow 0 < \begin{matrix} a_n \\ b_n \\ c_n \end{matrix} < \delta \Rightarrow \begin{matrix} f(a_n) = 0 \\ f(b_n) = 1 \\ f(c_n) = -1 \end{matrix} \not\Rightarrow \left| f\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} - L \right| < \frac{1}{2}$ (any $L \in \mathbb{R}$)

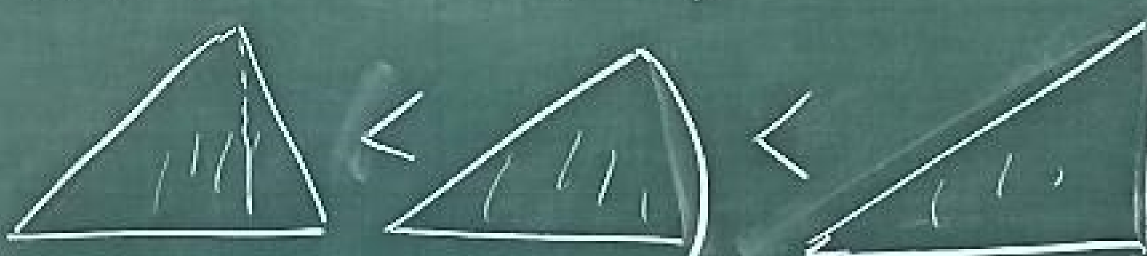
$\therefore$ when $\varepsilon = \frac{1}{2}$, corresponding δ does not exist

the lim does not exist

Limits involving $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ (=?)

Sol: We start with $\lim_{\theta \to 0^+}$

$\left(0<\theta<\frac{\pi}{2}\right)$



$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta \to 0^+} \cos\theta = 1 = \lim_{\theta \to 0^+} 1$

From Sandwich theorem (one sided version)

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = \underline{1}$$

What if $\frac{-\pi}{2} < \theta < 0$?

Let $\theta = -\varphi$. $0 < \varphi < \frac{\pi}{2}$

$\Rightarrow 1 > \frac{\sin\varphi}{\varphi} > \cos\varphi$

$\sin\varphi = -\sin\theta$, $\cos\varphi = \cos\theta$

$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$ also on $\frac{-\pi}{2} < \theta < 0$

From Sandwich Thm $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(since lim cos(theta)=1 and lim 1 = 1)