Suppose $f(x)$ is defined on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

Def: $\lim\limits_{x \to c} f(x) = L$

if for every $\varepsilon > 0$, there exists

(對每個正數ε, 皆存在)

a corresponding $\delta > 0$, such that

(一個對應的正數δ, 使得下式成立)

"$0 < |x-c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$" (1)

("若(x滿足) $0 < |x-c| < \delta$, 則 $|f(x)-L| < \varepsilon$")

Remark:

If we replace (1) by

" $|x-c|<\delta \Longrightarrow |f(x)-L|<\varepsilon$ " (1')

the result is different.

$(1') = (1) + $ " $|f(c)-L|<\varepsilon$ "

(for every $\varepsilon>0$)

That is,

$(1') = (1) + $ " $|f(c)-L|=0$ "

$(1') \Leftrightarrow$ " $\lim\limits_{x\to c} f(x)=L$ " + " $f(c)=L$ "

$\underset{\text{(later)}}{\Longleftrightarrow}$ $f(x)$ is cont. $x=c$

The counter example below shows that (2) cannot be a correct definition for (*) since the example satisfies (2), but not (*).

Wrong Statement:

$$\lim_{x \to c} f(x) = L \qquad (*)$$

if $f(x)$ gets closer to L

as x approaches c

i.e. $|x_2 - c| < |x_1 - c|$

$$\Rightarrow |f(x_2) - L| < |f(x_1) - L| \quad \cdots (2)$$

Counter example: $f(x) = x^2 + 1$, $c = 0$, $L = 0$

$f(x)$ satisfies (2), but $\lim_{x \to c} f(x) \neq L$

How to prove $\lim_{x \to c} f(x) = L$?

Ans: For any $\varepsilon > 0$, describe how to find corresponding $\delta > 0$

Eg 1 Use definition to show that

$$\lim_{x \to 0} x^3 = 0$$

Pf: For $\varepsilon > 0$, Want to find $\delta > 0$ such that

"$0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$"

$$|x^3-0|<\varepsilon$$

$$\Leftrightarrow -\varepsilon < x^3-0 < \varepsilon$$

$$\Leftrightarrow -\varepsilon^{\frac{1}{3}} < x < \varepsilon^{\frac{1}{3}}$$

$$\Leftrightarrow |x| < \varepsilon^{\frac{1}{3}}$$

$$\Leftarrow 0 < |x-0| < \varepsilon^{\frac{1}{3}}$$

Take $\delta = \varepsilon^{\frac{1}{3}}$, then

$$0<|x-0|<\delta \Rightarrow |x^3-0|<\varepsilon$$

$\#$

Eg2: Show that $\lim\limits_{x\to 5}\sqrt{x-1}=2$

Pf: For every $\varepsilon>0$ ($\underline{0<\varepsilon<4}$)

$$|\sqrt{x-1}-2|<\varepsilon$$

$$\Leftrightarrow -\varepsilon<\sqrt{x-1}-2<\varepsilon$$

$$\Leftrightarrow (2-\varepsilon)^2<x-1<(2+\varepsilon)^2$$

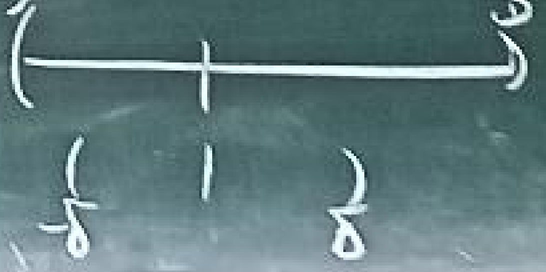
$$\Leftrightarrow (2-\varepsilon)^2-4<x-5<(2+\varepsilon)^2-4$$

(We demostrate by $\varepsilon=1$)

$$-3<x-5<5$$

take $\delta=\min(3,5)=3$

$$\Leftarrow 0<|x-5|<\delta$$



The method of finding delta for (epsilon=1) on previous page also works for general 0 < epsilon < 4

For general $0<\varepsilon<4$

$$(2-\varepsilon)^2-4 < x-5 < (2+\varepsilon)^2-4$$

"$-$" if $0<\varepsilon<4$ "$+$"

Take $\delta = \min\big(\underbrace{4-(2-\varepsilon)^2}_{\text{"}+\text{"}},\ \underbrace{(2+\varepsilon)^2-4}_{\text{"}+\text{"}}\big) > 0$

$\Leftarrow\ 0<|x-5|<\delta$

Remark:

In general, we only need to find $\delta>0$ for every ε in $0<\varepsilon<\varepsilon_0$

$\because$ If $\varepsilon_1<\varepsilon_2$ and "$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon_1$"

$\Rightarrow$ "$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon_2$"