Chap 2 Limit and Continuity

Informal definition: (p81)

Suppose $f(x)$ is defined on $(c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. We say $\lim_{x \to c} f(x) = L$ if $f(x)$ is arbitrarily close to L when x is sufficiently close to c.

Eg1: $\lim_{x \to 0} f(x)$ does not exist

(note: $f(0)$ is not relevant)

(i) $f(x) = \frac{x}{|x|}$, $x \neq 0$ (jump at $x=0$)

(ii) $f(x) = \frac{1}{x}$ or $\frac{1}{x^2}$, $x \neq 0$ (diverges to $\pm\infty$, unbounded near 0)

(iii) $f(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ (bounded, but oscillatory near 0)

$= \begin{cases} 0, & x = \frac{1}{n\pi} \\ \pm 1, & x = \frac{1}{(2n \pm \frac{1}{2})\pi} \end{cases}$ (p82) (graph)

Eg 2 (a) $f(x)=x$, $\lim\limits_{x\to c} f(x) = c$

(b) $f(x)=k$ $\qquad\qquad\qquad k$

Thm 1 (Limit laws)

If $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$

Then ① ②: $\lim\limits_{x\to c}(f(x) \pm g(x)) = L+M$

③ $\lim\limits_{x\to c}(k f(x)) = k\cdot L$

④ $\lim\limits_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

⑤ $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

⑥ $\lim\limits_{x\to c}(f(x)^n) = L^n$ (n = positive integer)

⑦ $\lim\limits_{x\to c} f(x)^{\frac{1}{n}} = L^{\frac{1}{n}}$ if $L>0$

Eg 3

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ (Limit of a polynomial)

$= c^3 + 4c^2 - 3$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ (Limit of a rational function)

$= \dfrac{c^4 + c^2 - 1}{c^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{13}$

("polynomial" + Limit law 7)

Limits involving quotient

$\left(\text{"}\frac{0}{0}\text{"}\right)$ or $\left(\text{"}\frac{\text{not } 0}{0}\text{"}\right)$

Eg 4. $\lim\limits_{x \to 1} \frac{x^2+x-2}{x^2-x}$

$x^2-x=0$
$x^2+x-2=0$ when $x=1$

$\Rightarrow x^2-x=(x-1)\cdot x$
$x^2+x-2=(x-1)(x+2)$

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}\cdot x} = 3$$

Eg5 $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{x\to 0} \frac{(\cancel{x^2}+100-10^2)}{\cancel{x^2}(\sqrt{x^2+100}+10)}$$

$$= \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{20}$$

Eg 6. $\lim_{x\to 1} \frac{x^2-1}{(x-1)^2} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-1)}$

(diverges) does not exist.

§ Sandwich Theorem

<u>Thm 4</u>. Suppose $g(x) \le f(x) \le h(x)$

on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

Then $\lim_{x\to c} f(x) = L$

Eg 7 $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$

$\because -1 \leq \sin\frac{1}{x} \leq 1$

$\therefore -|x| \leq x\sin\frac{1}{x} \leq |x| \quad x \neq 0$

$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|$

$\xrightarrow{\text{Sandwich thm}} \lim_{x \to 0} x\sin\frac{1}{x} = 0$

Precise definition of limit

Def: $\lim\limits_{x \to c} f(x) = L$

if for every $\varepsilon > 0$,

there exists a corresponding $\delta > 0$

such that

"$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$"