Chap 2 Limit and Continuity

Informal definition: (p81)

Suppose $f(x)$ is defined on $(c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. We say $\lim_{x \to c} f(x) = L$ if $f(x)$ is arbitrarily close to L when x is sufficiently close to c.

Eg1: $\lim_{x \to 0} f(x)$ does not exist

(note: $f(0)$ is not relevant)

(i) $f(x) = \frac{x}{|x|}$; $x \neq 0$ $\left(\begin{array}{l}\text{jump} \\ \text{at } x=0\end{array}\right)$

(ii) $f(x) = \frac{1}{x}$ or $\frac{1}{x^2}$ $\left(\begin{array}{l}\text{diverges to } \pm\infty \\ \text{unbounded near } 0\end{array}\right)$

$x \neq 0$

(iii) $f(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ $\left(\begin{array}{l}\text{bounded, but} \\ \text{oscillatory near } 0\end{array}\right)$

$= \begin{cases} 0, & x = \frac{1}{n\pi} \\ \pm 1, & x = \frac{1}{(2n \pm \frac{1}{2})\pi} \end{cases}$ (p82) (graph)

Eg 2 (a) $f(x) = x$, $\lim\limits_{x \to c} f(x) = c$.

(b) $f(x) = k$ $\qquad k$

Thm 1 (Limit laws)

If $\lim\limits_{x \to c} f(x) = L$, $\lim\limits_{x \to c} g(x) = M$

then ① ②: $\lim\limits_{x \to c} (f(x) \pm g(x)) = L \pm M$

③ $\lim\limits_{x \to c} (k f(x)) = k \cdot L$

④ $\lim\limits_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

⑤ $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

⑥ $\lim\limits_{x \to c} (f(x)^n) = L^n$ (n = positive integer)

⑦ $\lim\limits_{x \to c} f(x)^{\frac{1}{n}} = L^{\frac{1}{n}}$ if $L > 0$

Eg 3

ⓐ $\lim_{x \to c} (x^3 + 4x^2 - 3)$ (Limit of a polynomial)

$= c^3 + 4c^2 - 3$

ⓑ $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ (Limit of a rational function)

$= \dfrac{c^4 + c^2 - 1}{c^2 + 5}$

ⓒ $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{13}$

("polynomial" + Limit law 7)

Limits involving quotient

$\left(\text{"}\frac{0}{0}\text{"}\right)$ or $\left(\frac{\text{"not } 0\text{"}}{0}\right)$.

Eg 4. $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$

$x^2-x=0$

$x^2+x-2=0$ when $x=1$

$\Rightarrow x^2-x=(x-1)\cdot x$

$x^2+x-2=(x-1)(x+2)$

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}\cdot x} = 3$$

Eg5 $\lim_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2}$

$= \lim_{x\to 0} \dfrac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$

$(a-b)(a+b)=a^2-b^2$

$= \lim_{x\to 0} \dfrac{\cancel{(x^2+100-10^2)}}{\cancel{x^2}(\sqrt{x^2+100}+10)}$

$= \dfrac{1}{\sqrt{0^2+100}+10} = \dfrac{1}{20}$

Eg 6. $\lim_{x \to 1} \frac{x^2-1}{(x-1)^2} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-1)}$ does not exist

(diverges)

§ Sandwich Theorem

Thm 4. Suppose $g(x) \leq f(x) \leq h(x)$

on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

If $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

Then $\lim_{x \to c} f(x) = L$

Eg 7 $\lim_{x\to 0} x\cdot\sin\left(\frac{1}{x}\right)$

$\because -1 \leq \sin\frac{1}{x} \leq 1$

$\therefore -|x| \leq x\sin\frac{1}{x} \leq |x|, \quad x \neq 0$

$\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0}|x|$

Sandwich thm $\Longrightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$

Precise definition of limit

Def: $\lim\limits_{x \to c} f(x) = L$

if for every $\varepsilon > 0$,

there exists a corresponding $\delta > 0$

such that

$$\text{"}0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon\text{"}$$