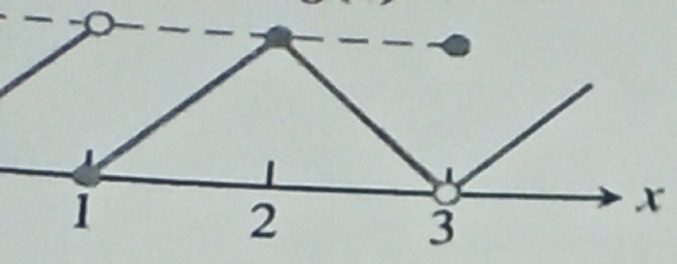
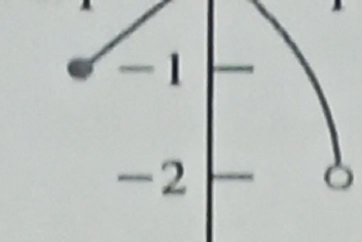
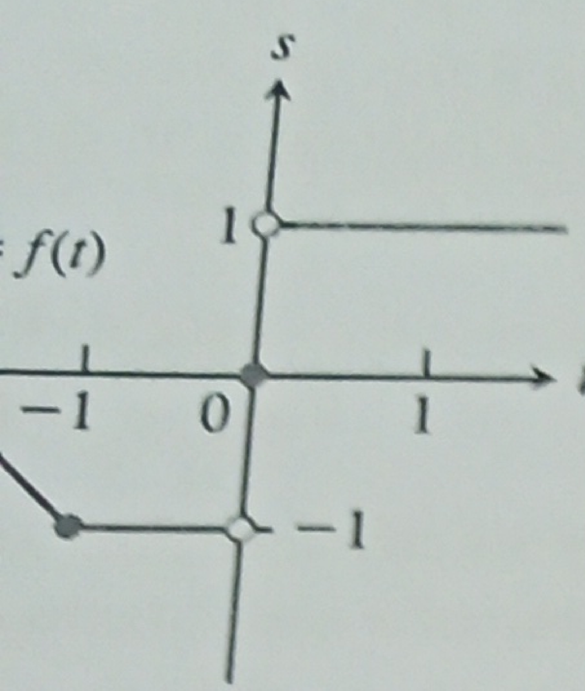




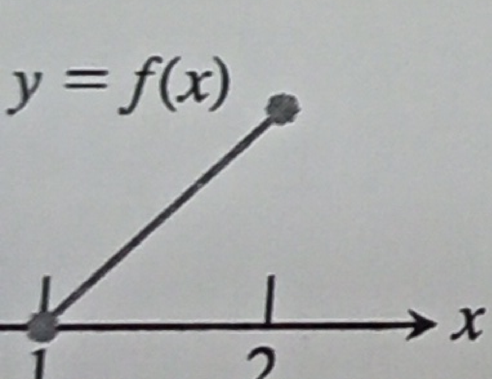
aphed here, find the following limits or exist.

(t) **c.** $\lim_{t\to 0} f(t)$ **d.** $\lim_{t\to -0.5} f(t)$



ements about the function $y = f(x)$ hich are false?

nt c in $(-1, 1)$.



ts about the function $y = f(x)$ re false?

Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$ **6.** $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to -3} (x^2 - 13)$ **12.** $\lim_{x\to 2} (-x^2 + 5x - 2)$

13. $\lim_{t\to 6} 8(t-5)(t-7)$ **14.** $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to 2} \dfrac{2x+5}{11 - x^3}$ **16.** $\lim_{s\to 2/3} (8 - 3s)(2s - 1)$

17. $\lim_{x\to -1/2} 4x(3x+4)^2$ **18.** $\lim_{y\to 2} \dfrac{y+2}{y^2 + 5y + 6}$

19. $\lim_{y\to -3} (5 - y)^{4/3}$ **20.** $\lim_{z\to 4} \sqrt{z^2 - 10}$

21. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1} + 1}$ **22.** $\lim_{h\to 0} \dfrac{\sqrt{5h+4} - 2}{h}$

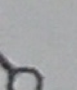
Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to 5} \dfrac{x-5}{x^2 - 25}$ **24.** $\lim_{x\to -3} \dfrac{x+3}{x^2 + 4x + 3}$

25. $\lim_{x\to -5} \dfrac{x^2 + 3x - 10}{x+5}$ **26.** $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x-2}$

27. $\lim_{t\to 1} \dfrac{t^2 + t - 2}{t^2 - 1}$ **28.** $\lim_{t\to -1} \dfrac{t^2 + 3t + 2}{t^2 - t - 2}$

29. $\lim_{x\to -2} \dfrac{-2x - 4}{x^3 + 2x^2}$ **30.** $\lim_{y\to 0} \dfrac{5y^3 + 8y^2}{3y^4 - 16y^2}$



ref

31. $\lim_{x \to 1} \frac{x^{-1} - 1}{x - 1}$

32. $\lim_{x \to 0} \frac{\frac{1}{x - 1} + \frac{1}{x + 1}}{x}$

33. $\lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1}$

34. $\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16}$

35. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

36. $\lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}}$

37. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$

38. $\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

39. $\lim_{x \to 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$

40. $\lim_{x \to -2} \frac{x + 2}{\sqrt{x^2 + 5} - 3}$

41. $\lim_{x \to -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$

42. $\lim_{x \to 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x \to 0} (2 \sin x - 1)$

44. $\lim_{x \to \pi/4} \sin^2 x$

45. $\lim_{x \to 0} \sec x$

46. $\lim_{x \to \pi/3} \tan x$

47. $\lim_{x \to 0} \frac{1 + x + \sin x}{3 \cos x}$

48. $\lim_{x \to 0} (x^2 - 1)(2 - \cos x)$

49. $\lim_{x \to -\pi} \sqrt{x + 4} \cos (x + \pi)$

50. $\lim_{x \to 0} \sqrt{7 + \sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

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Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x) = x^2, \quad x = 1$
58. $f(x) = x^2, \quad x = -2$
59. $f(x) = 3x - 4, \quad x = 2$
60. $f(x) = 1/x, \quad x = -2$
61. $f(x) = \sqrt{x}, \quad x = 7$
62. $f(x) = \sqrt{3x + 1}, \quad x = 0$

Using the Sandwich Theorem

63. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x \to 0} f(x)$.

64. If $2 - x^2 \le g(x) \le 2 \cos x$ for all x, find $\lim_{x \to 0} g(x)$.

65. **a.** It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{x \sin x}{2 - 2 \cos x}?$$

Give reasons for your answer.

T **b.** Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2 \cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

xercises 67–76.

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$_{-1} f(x)$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 1$.

76. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \to 2} f(x) = 0$? Give reasons for your answers.

79. If $\displaystyle\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x \to 4} f(x)$.

80. If $\displaystyle\lim_{x \to -2} \frac{f(x)}{x^2} = 1$, find

a. $\displaystyle\lim_{x \to -2} f(x)$

b. $\displaystyle\lim_{x \to -2} \frac{f(x)}{x}$

81. a. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 3$, find $\lim_{x\to 2} f(x)$.

b. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 4$, find $\lim_{x\to 2} f(x)$.

82. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$

b. $\lim_{x\to 0} \dfrac{f(x)}{x}$

T 83. a. Graph $g(x) = x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T 84. a. Graph $h(x) = x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the followin

a. Plot the function near the point c being approa

b. From your plot guess the value of the limit.

85. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

86. $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x\to 0} \dfrac{\sqrt[3]{1 + x} - 1}{x}$

88. $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x\to 0} \dfrac{1 - \cos x}{x \sin x}$

90. $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. W like "gets arbitrarily close to" in the informal definition with sp be applied to any particular example. With a precise definition standings, prove the limit properties given in the preceding se important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L gap between $f(x)$ and L can be made "as small as we choose" to c. Let us see what this would require if we specified the si and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1)$ $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 b