Brief solutions to Quiz 10

June 09, 2022:

1. (30 pts) Suppose $\boldsymbol{F}(x,y) = (M(x,y), N(x,y))$, both M and N are continuously differentiable (i.e. 1st derivatives are all continuous) on $\mathbb{R}^2$. Let C be a simple closed curve and $\mathcal{R}$ the region enclosed by C. State both forms of Green's Theorem (need not repeat the statements above).

   Ans:

   See Theorem 4 and Theorem 5 on Section 16.4 of the textbook.

2. (30 pts) Let $\boldsymbol{F}(x,y) = (x+y+2, 2x+y-1)$. Use any method or Theorem (including the formula of area of an ellipse) to evaluate $\displaystyle\oint_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ where C is the ellipse $\{\dfrac{x^2}{4}+y^2 = 1\}$.

   Ans:

   From Green's Theorem in tangential form:

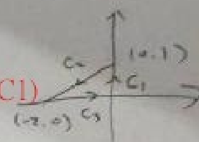
   $$\oint_{\frac{x^2}{4}+y^2=1} \boldsymbol{F}\cdot\boldsymbol{T}\,ds = \oint_{\frac{x^2}{4}+y^2=1} M\,dx + N\,dy = \iint_{\frac{x^2}{4}+y^2<1} (N_x - M_y)\,dA \quad (\textbf{20 pts})$$

   Since $N_x - M_y = 1$, area of the ellipse $= ab\pi = 2\pi$. Answer $= 2\pi$. (**10 pts**)

3. (40 pts) Prove Green's Theorem for $\boldsymbol{F}(x,y) = (M(x,y), 0)$, M is continuously differentiable, in either form (but not both) on the region $\mathcal{R}$ enclosed by $x = 0$, $y = 0$ and $\frac{x}{-2} + y = 1$ in the 2nd quadrant. That is, the triangle with vertices $(-2,0)$, $(0,0)$ and $(0,1)$. Show all details including the parametrizations you choose.

Ans:

tan: $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$



C_1: $x = 0$, $y = t$, $0 \le t \le 1$

C_2: $x = -t$, $y = 1 - \frac{t}{2}$, $0 \le t \le$

C_3: $x = t - 2$, $y = 0$, $0 \le t \le$

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(Mdx+Ndy= 0 on C1)

LHS $=$ 8 $\underbrace{\int_0^2 -M(-t, 1-\tfrac{t}{2})\,dt}_{C_2} + \underbrace{\int_0^2 M(t-2, 0)\,dt}_{C_3}$ (N=0) (N=0)

$u = -t$, $v = t-2$: $=$ 8 $-\int_{-2}^{0} M(u, 1+\tfrac{u}{2})\,du + \int_{-2}^{0} M(v, 0)\,dv$

RHS $=$ 8 $\int_{-2}^{0}\int_{0}^{1+\frac{x}{2}} -M_y(x,y)\,dy\,dx$

$=$ 8 $-\int_{-2}^{0} M(x, 1+\tfrac{x}{2})\,dx + \int_{-2}^{0} M(x, 0)\,dx$

$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

norm: $\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$

C_1, C_3 same

C_2: $x = 2(1-t) - 2$, $y = 1 - t$, $0 \le t \le$

8

LHS $=$ 8 $\oint_{C_1} M\,dy + \oint_{C_2} M\,dy$ (N=0) (N=0) (Mdy-Ndx=0 on C3)

$u = t$, $v = 1-t$: $=$ 8 $\int_0^1 M(0,t)\,dt - \int_0^1 M(-2t, 1-t)\,dt = \int_0^1 M(0,u)\,du - \int_0^1 M(2v-2, v)$

RHS $=$ 8 $\int_0^1 \int_{2y-2}^{0} M_x(x,y)\,dx\,dy$

$=$ 8 $\int_0^1 M(0,y)\,dy - \int_0^1 M(2y-2, y)\,dy$