Brief solutions to Quiz 2

Mar 08, 2022:

1. (10 pts + 20 pts) Write down the definition of $\sum_{n=1}^{\infty} a_n = L$, $(L \neq \pm\infty)$ and use it to show that

$$\sum_{n=1}^{\infty} a_n = L \implies \lim_{n\to\infty} a_n = 0.$$

Both parts will involve the partial sum $s_n = a_1 + a_2 + \cdots + a_n$.

Ans:

Definition: $\sum_{n=1}^{\infty} a_n = L$ if $\lim_{n\to\infty} s_n = L$.

Proof: See textbook (Section 10.2, Theorem 7) or the lecture note.

2. (10 pts + 20 pts) Write down the statement of The Integral Test (need not prove it) and use it to derive the convergence/divergence of $\sum_{n=1}^{\infty} n^{-p}$ for $p > 0$. For this problem, you need to show detail computation of the corresponding improper integral (as a limit).

Ans:

The Integral Test: See textbook (Section 10.3, Theorem 9) or the lecture note.

Convergence/divergence of the p-series: See textbook (Section 10.3, Example 3) or the lecture note.

Correct statement: 10 pts. Correct answer on convergence/divergence: 10 pts. Correct explanation: 10 pts.

3. (20 pts + 20 pts) Determine the convergence/divergence of

$$(a): \sum_{n=2}^{\infty} \frac{1}{\ln n} \quad (b): \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$$

For this problem, you may use the convergence/divergence of elementary series such as geometric series, p series without proving it again.

Ans:

(a): divergent. Use The Comparison Test, compare a_n with $\frac{1}{n}$: $\sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n}$

Correct answer: 10 pts, correct explanation: 10 pts.

(b): convergent. Use Limit Comparison Test, compare a_n with $\frac{1}{n^2}$: Use L'Hôpital's Rule to show that $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{\frac{-1}{n^3}}{1+\frac{1}{n^2}}}{\frac{-1}{n^3}} = 1.$

Correct answer: 10 pts, correct explanation with details: 10 pts.