Brief solutions to Quiz 1

Mar 01, 2022:

1. (33 pts) Write down the definition of $\int_0^{\frac{\pi}{2}} \tan x \, dx$ (as a limit) and use any method to show whether it converges or diverges.

   Ans:

   $$\int_0^{\frac{\pi}{2}} \tan x \, dx = \lim_{b \to \frac{\pi}{2}^-} \int_0^b \tan x \, dx = \lim_{b \to \frac{\pi}{2}^-} -\ln(\cos x)\Big|_0^b = +\infty, \quad \text{divergent}$$

   Another way to show divergence: Since

   $$\lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{\left(\frac{\pi}{2} - x\right)^{-1}} = 1 \quad \text{(Use L'Hôpital's Rule)}$$

   and

   $$\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^{-1} dx = +\infty \quad \text{(See page 13 of mse22s\_l01.pdf, } p = 1\text{)}$$

   By type II analogue of the Limit Comparison Test, $\int_0^{\frac{\pi}{2}} \tan x \, dx$ diverges.

   Definition: 13 pts. Correct answer on convergence/divergence: 15 pts. Correct explanation: 5 pts.

2. (33 pts) Write down the definition of $\int_1^{\infty} \frac{3 + \cos x}{x} dx$ (as a limit) and use any method to show whether it converges or diverges.

   Ans:

   Definition:

   $$\int_1^{\infty} \frac{3 + \cos x}{x} dx = \lim_{b \to \infty} \int_1^b \frac{3 + \cos x}{x} dx$$

   Test convergence:

   $$-1 \le \cos x \le 1 \implies \frac{2}{x} \le \frac{3 + \cos x}{x}$$

   Moreover

   $$\int_1^{\infty} \frac{2}{x} \, dx = \infty \quad (p = 1)$$

   By Direct Comparison Test, $\int_1^{\infty} \frac{3 + \cos x}{x} \, dx = \infty$ diverges.

   Definition: 13 pts. Correct answer on convergence/divergence: 15 pts. Correct explanation: 5 pts.

3. (34 pts) Write down the definition of $\lim_{n\to\infty} a_n = L$ and use any method to show whether $\lim_{n\to\infty} \frac{n!}{n^n}$ converges or diverges.

Ans:

Definition:

See page 588 of the textbook or page 8 of mse22s_l02.pdf.

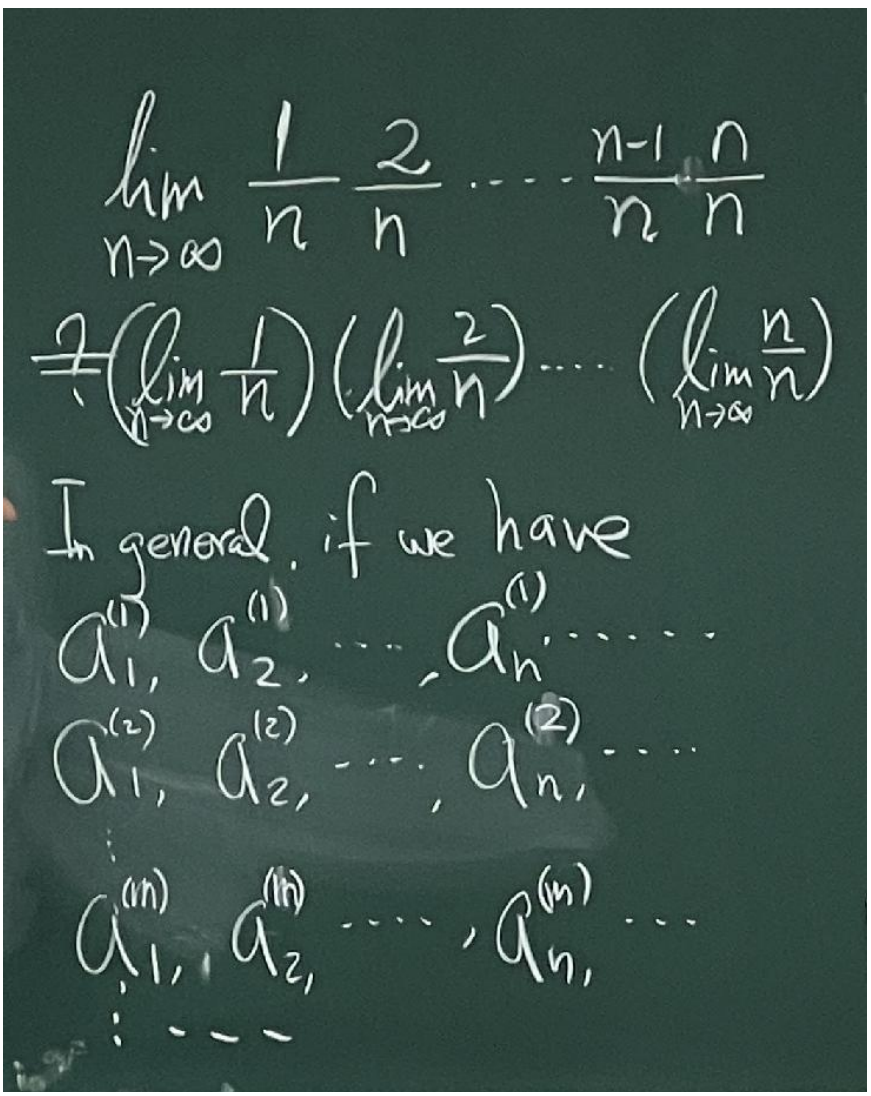
Test convergence:

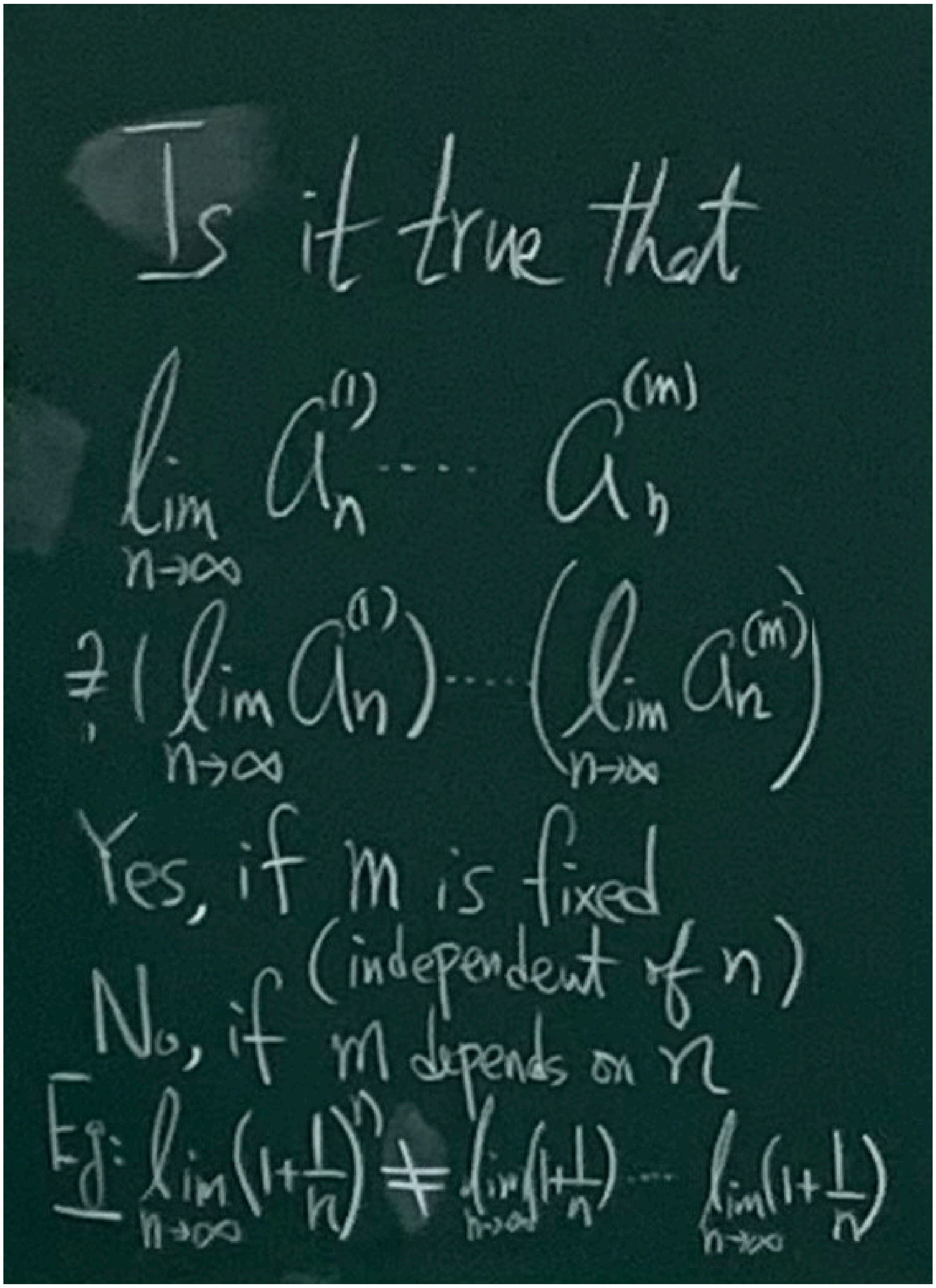
See page 5 of homework 01 solution.

Answer: it converges (to 0).

Definition: 14 pts. Correct answer on convergence/divergence: 15 pts. Correct explanation: 5 pts.

A common mistake:




Is it true that
$\lim_{n\to\infty} a_n^{(1)} \cdots\cdots a_n^{(m)}$
$\stackrel{?}{=} \left(\lim_{n\to\infty} a_n^{(1)}\right) \cdots\cdots \left(\lim_{n\to\infty} a_n^{(m)}\right)$
Yes, if m is fixed
(independent of n)
No, if m depends on n
Eg: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \neq \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \cdots \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$