Brief solutions to Final Exam

1. (15 pts) Evaluate $I = \int_0^\infty e^{-x^2} dx$.

   Hint: $I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$.

   Answer.

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dxdy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta \quad (\textbf{10pts}).$$

   Therefore

$$I = \left(\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta \right)^{\frac{1}{2}} = \left(\int_0^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-r^2} rdr \right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{2} \quad (\textbf{5pts}).$$

2. (15 pts) Replace

$$\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-z^2}} rdr\ dz\ d\theta$$

   by a triple integral in spherical coordinates and find its value by any one of the two integrals of your choice.

   Answer.

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi\, d\rho d\phi\, d\theta = \int_0^2 \rho^2\, d\rho \int_0^{\frac{\pi}{2}} \sin\phi\, d\phi \int_0^{2\pi} d\theta \quad (\textbf{10 pts}) \ = \frac{16\pi}{3} \quad (\textbf{5 pts})$$

3. (15 pts) True or false? Give details.

   If $f(x,y,z)$ has continuous first derivatives in a domain $\mathcal{D}$, and $\mathcal{C} = \{(x(t), y(t), z(t)),\ 0 \le t \le 1\}$ is a smooth curve in $\mathcal{D}$. Then $\int_\mathcal{C} \nabla f \cdot \boldsymbol{T}\ ds$ depends only on f, $(x(0), y(0), z(0))$ and $(x(1), y(1), z(1))$.

   Answer. True. **(3 pts)**

$$\begin{aligned} & \int_C \nabla f \cdot \mathbf{T}\, ds \\ = & \int_0^1 (f_x(x(t),y(t),z(t))x'(t) + f_y(x(t),y(t),z(t))y'(t) + f_z(x(t),y(t),z(t))z'(t))\, dt \ \textbf{(4 pts)} \\ = & \int_0^1 \partial_t(f(x(t),y(t),z(t))\, dt \ \textbf{(4 pts)} = f(x(1),y(1),z(1)) - f(x(0),y(0),z(0)). \ \textbf{(4 pts)} \end{aligned}$$

4. (15 pts) Let $\boldsymbol{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$, $\mathcal{C} = \{\frac{x^2}{9} + \frac{y^2}{4} = 1\}$. Evaluate $\oint_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{T}\, ds$.

Answer.

See Lecture 27, page 5-7.

Check that $\boldsymbol{F}$ satisfies the component test (**5 pts**).

Use Green's Theorem to show that $\oint_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \oint_{\mathcal{C}_\varepsilon} \boldsymbol{F} \cdot \boldsymbol{T}\, ds$ where $\mathcal{C}_\varepsilon$ is a circle of radius ε centered at the origin with ε small enough (**5 pts**).

Evaluate (with all details) to get $\oint_{\mathcal{C}_\varepsilon} \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 2\pi$ (**5 pts**).

5. (15 pts) Evaluate the surface area of $\mathcal{S} = \{z = \sqrt{x^2+y^2},\ 1 \le xy \le 2,\ 1 \le x/y \le 3,\ x > 0,\ y > 0\}$.

Answer.

Let $f(x,y) = \sqrt{x^2+y^2}$. We have

$$f_x = \frac{x}{\sqrt{x^2+y^2}},\ f_y = \frac{y}{\sqrt{x^2+y^2}}.\ \textbf{(2 pts)}$$

and therefore the surface area is given by

$$|\mathcal{S}| = \iint_{\mathcal{R}} \sqrt{f_x^2 + f_y^2 + 1}\, dxdy\ (\textbf{2pts}) = \iint_{\mathcal{R}} \sqrt{2}\, dxdy\ (\textbf{1pt})$$

where $\mathcal{R} = \{1 \le xy \le 2,\ 1 \le x/y \le 3,\ x > 0,\ y > 0,\ z = 0\}$.

Let $u = xy$, $v = \frac{x}{y}$, $x, y > 0$ **(2 pts)**. Then $\mathcal{R} = \{1 \le u \le 2, 1 \le v \le 3\}$ **(2 pts)** and the Jacobian is given by

$$\left\| \begin{matrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\frac{\sqrt{u}}{v\sqrt{v}} \end{matrix} \right\| = \left| -\frac{1}{2v} \right| = \frac{1}{2v}.\ \textbf{(2 pts)}$$

Thus, the surface area can be evaluated by

$$|\mathcal{S}| = \int_1^2 \int_1^3 \frac{\sqrt{2}}{2v}\, dvdu\ \textbf{(2 pts)} = \frac{\ln 3}{\sqrt{2}}.\ \textbf{(2 pts)}$$

6. (30 pts) Let $\mathcal{D} = \{x^2+y^2+z^2 < 4, z > 0\}$ (upper half of a ball with radius 2), $\mathcal{S} = \{x^2+y^2+z^2 = 1, z > 0\}$ (an open surface) and $\boldsymbol{F}(x,y,z) = (y, -x, 1)$.

   (a) Which of $\mathcal{S}$ and $\mathcal{D}$ can you apply Stokes' Theorem? State Stokes' Theorem and verify it with this $\boldsymbol{F}$. That is, compute integrals on both sides of the Stokes' Theorem and check the two numbers are the same.

   (b) Do the same for Divergence Theorem.

Answer.

(a) State Stokes' Theorem correctly on $\mathcal{S}$: (**5 pts**).

Correct evaluation (with all details) of line integral and surface integral in Stokes' Theorem ($= -2\pi$): (**5 pts**) each.

(b) State Divergence Theorem correctly on $\mathcal{D}$: (**5 pts**).

Correct evaluation (with all details) of volume integral and surface integral in Divergence Theorem ($= 0$): (**5 pts**) each.