Brief solutions to Midterm 2 (v02)

1. (10 pts) Evaluate

$$\frac{d}{dy}\int_1^{2+y^2}\frac{\cos(xy)}{x}dx$$

Ans:

$$= \frac{\cos(xy)}{x}|_{x=2+y^2}\cdot\frac{d}{dy}(2+y^2)+\int_1^{2+y^2}\frac{d}{dy}\frac{\cos(xy)}{x}dx \quad \textbf{(6 pts)}$$

$$= \frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y+\int_1^{2+y^2}-\sin(xy)dx \quad \textbf{(2 pts)}$$

$$= \frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y+\frac{1}{y}\Big(\cos((2+y^2)y)-\cos y\Big) \quad \textbf{(2 pts)}$$

2. (10 pts) Find the equation of plane normal to the following curve at $(1,-1,1)$

$$\begin{cases} x^2+2y^2+3z^2=6 \\ x+y+z=1 \end{cases}$$

Ans:

First compute the two gradients at $(1,-1,1)$

$$(2x,4y,6z)_{(1,-1,1)}=(2,-4,6)\ \textbf{(2 pts)}$$

and

$$(1,1,1)|_{(1,-1,1)}=(1,1,1).\ \textbf{(2 pts)}$$

The normal vector of the plane is parallel to the outer product of these two gradients:

$$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}=(5,-2,-3).\ \textbf{(2 pts)}$$

Therefore, the equation of the plane is

$$5(x-1)-2(y+1)-3(z-1)=0.\ \textbf{(4 pts)}$$

3. (10 pts) Find absolute maxima and minima of $f(x,y)=x^2+xy+y^2-6x$ on the rectangular domain $0\le x\le 5$, $-3\le y\le 0$.

Ans:

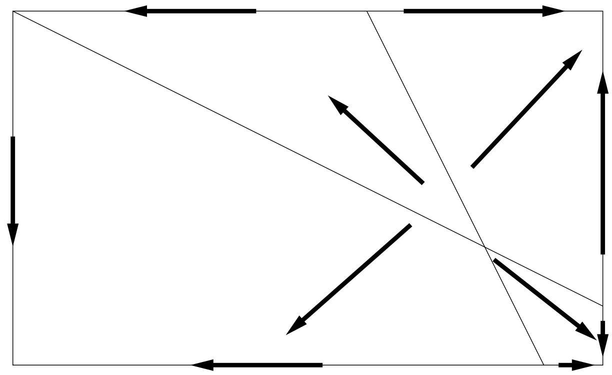

Figure 1: The gradient analysis for problem 5.

Method I:

First find the gradient:

$$\nabla f(x, y) = (2x + y - 6, x + 2y).$$

Therefore one can plot the gradients as in figure 1 (**5 pts**).

From the plot it is easy to see that $f(4, -2) = -12$ is the only local minimum since there is no local minimum on the boundary. Therefore $f(4, -2) = -12$ is also the absolute minimum **(2 pts)**.

Moreover, the local maximum consists of the three corners $(0, -3)$, $(5, -3)$ and $(5, 0)$. Upon comparing the values of f on the corners $(0, -3)$, $(5, -3)$ and $(5, 0)$ it follows that the absolute maxima is $f(0, -3) = 9$ **(3 pts)**.

Method II:

Find all critical points in

interior: $f(4, -2) = -12$. (**2 pts**)

edges: $f(3, 0) = -9$, $f(\frac{9}{2}, -3) = -\frac{45}{4}$, $f(5, -\frac{5}{2}) = -\frac{45}{4}$. (**4 pts**)

corners: $f(0, 0) = 0$, $f(5, 0) = -5$, $f(0, -3) = 9$, $f(5, -3) = 11$. (**2 pts**)

2 pts below will be given only when all above are correctly computed:

Absolute minimum: $f(4, -2) = -12$. Absolute maximum: $f(0, -3) = 9$. (**2 pts**)

4. (10 pts) Use Lagrangian multipliers (and only Lagrangian multipliers) to find extreme values of $f(x, y, z) = xy + 2z^2$ on

$$\begin{cases} x^2 + y^2 + z^2 = 9 \\ x - y = 0 \end{cases}$$

Ans: Let $g_1(x, y, z) = x^2 + y^2 + z^2 - 9$, $g_2(x, y, z) = x - y$.

Solve from

$$\begin{cases} g_1(x,y,z) = 0, \\ g_2(x,y,z) = 0, \\ \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \end{cases} \qquad (\mathbf{4\,pts})$$

Critical points: $(x,y,z) = (0,0,\pm 3)$, $\pm(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0)$. (**4 pts**)

2 pts below will be given only when all above are correctly computed:

Max: $f(0,0,\pm 3) = 18$. Min: $f(\pm(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0)) = \frac{9}{2}$. (**2 pts**)

5. (10 pts) Let $f(x,y) = x^3 + y^3$ and $g(r,\theta) = f(r\cos\theta, r\sin\theta)$. Evaluate

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}$$

Ans:

$$\frac{\partial g}{\partial r} = 3r^2\cos^3\theta + 3r^2\sin^3\theta, \qquad (\mathbf{2\ pts})$$

$$\frac{\partial^2 g}{\partial r^2} = \frac{\partial}{\partial r}(\frac{\partial g}{\partial r}) = 6r\cos^3\theta + 6r\sin^3\theta, \qquad (\mathbf{2\ pts})$$

$$\frac{\partial g}{\partial \theta} = -3r^3\cos^2\theta\sin\theta + 3r^3\cos\theta\sin^2\theta, \qquad (\mathbf{2\ pts})$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial}{\partial \theta}(\frac{\partial g}{\partial \theta}) = 3r^3(3\sin^2\theta\cos\theta + 3\cos^2\theta\sin\theta - \cos\theta - \sin\theta). \qquad (\mathbf{2\ pts})$$

Therefore,

$$\begin{aligned} \frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} &= (6r\cos^3\theta + 6r\sin^3\theta) + (3r\cos^3\theta + 3r\sin^3\theta) \\ &\quad + 3r(3\sin^2\theta\cos\theta + 3\cos^2\theta\sin\theta - \cos\theta - \sin\theta) \\ &= 6r\cos\theta + 6r\sin\theta. \qquad (\mathbf{2\ pts}) \end{aligned}$$

6. (10 pts) Taylor's formula for $f(x,y)$: Assume all partial derivatives of any order of f are continuous. Find a quadratic approximation of $f(x,y) = \ln(2x+y+1)$ near the origin.

Ans:

Method I: Quadratic approximation:

$$f_x(x,y) = \frac{2}{1+2x+y}, \qquad f_y(x,y) = \frac{1}{1+2x+y}, \ (\mathbf{2\ pts})$$

$$f_{xx}(x,y) = \frac{-4}{(1+2x+y)^2}, \quad f_{xy}(x,y) = \frac{-2}{(1+2x+y)^2}, \quad f_{yy}(x,y) = \frac{-1}{(1+2x+y)^2} \ (\mathbf{3\ pts})$$

$$\begin{aligned} Q(x,y) &= f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2) \\ &= 0+2x+y-\frac{1}{2}(4x^2+4xy+y^2)\ \textbf{(5 pts)} \end{aligned}$$

Method II:

Let $z = 2z + y$ and consider Taylor's formula for $\ln(1+z)$. Since the quadrative approximation of $\ln(1+z)$ is $Q(z) = z - \frac{1}{2}z^2$. We see that

$$Q(x,y) = (2x+y) - \frac{1}{2}(2x+y)^2.$$

7. (10 pts) Evaluate $\left(\frac{\partial u}{\partial x}\right)_y$ at $(x,y,z,w) = (1,1,1,1)$ where $u(x,y,z,w) = x^2+y^2+z^2+w^2$ with the constraint $x+y+z+w=4$ and $x-y+z-w=0$.

Ans: First we compute z_x and w_z at $(1,1,1,1)$.

$$\begin{aligned} & 1+z_x+w_x = 0\ \textbf{(2 pts)} \\ & 1+z_x-w_x = 0\ \textbf{(2 pts)} \\ \Rightarrow\quad & z_x = -1, w_x = 0.\ \textbf{(2 pts)} \end{aligned}$$

Now we compute $\left(\frac{\partial u}{\partial x}\right)_y$ at $(1,1,1,1)$.

$$\left(\frac{\partial u}{\partial x}\right)_y = 2x+2zz_x+2ww_x|_{(1,1,1,1)}\ \textbf{(2 pts)} = 2-2=0.\ \textbf{(2 pts)}$$

8. (15 pts) Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$, for $(x,y) \neq (0,0)$ and $f(0,0)=0$. $P=(0,0)$ and $\boldsymbol{u}^\theta = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$.

   (a) Is f continuous at $(0,0)$? Explain.

   (b) For fixed θ, write down the definition of the directional derivative $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u}^\theta,P}$ and evaluate it.

   (c) Does f have a linear approximation at $(0,0)$? Explain.

Ans:

(a) (5 pts)
Let

$$x = r\cos\theta, y = r\sin\theta$$

it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = 0$$

therefore f is continuous at $(0,0)$.

(b) (5 pts)

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u}^\theta, P} = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0))}{h}$$

$$= \lim_{h\to 0} \frac{h(\cos^3\theta + \sin^3\theta)}{h} = \cos^3\theta + \sin^3\theta$$

(c) (5 pts)

Since $\nabla f(0,0)\cdot \boldsymbol{u}^\theta = (1,1)\cdot(\cos\theta, \sin\theta) \neq \cos^3\theta + \sin^3\theta = \left(\frac{df}{ds}\right)_{\boldsymbol{u}^\theta, P}$, f is not differentiable at $(0,0)$. Therefore f does not have a linear approximation.

9. Evaluate $\int_0^2 \int_y^2 \frac{\sin x}{x} dxdy$.

Ans: we have

$$\int_0^2 \int_y^2 \frac{\sin(x)}{x} dxdy$$

$$= \int_0^2 \int_0^x \frac{\sin(x)}{x} dydx \quad (\textbf{6 pts})$$

$$= \int_0^2 (\frac{\sin(x)}{x})(x)dx \quad (\textbf{2 pts})$$

$$= \int_0^2 \sin(x)dx = 1 - \cos 2 \quad (\textbf{2 pts})$$

10. (10 pts) Find all critical points of $f(x,y) = x^4 + y^4 + 4xy$ and determine whether they are local minima, local maxima or neither.

Ans:

$$f_x = 4x^3 + 4y, \quad f_y = 4y^3 + 4x, \quad (\textbf{1 pts})$$

$$(f_x, f_y) = (0,0) \implies (x,y) = (0,0), (1,-1), (-1,1) \quad (\textbf{2 pts})$$

$$f_{xx} = 12x^2,\ f_{xy} = 4,\ f_{yy} = 12y^2, \quad (\textbf{3 pts})$$

$$f_{xy}^2 - f_{xx}f_{yy} = 16, -128, -128 \text{ at } (0,0),\ (1,-1),\ (-1,1),\ \text{respectively.} \quad (\textbf{2 pts})$$

Therefore $f(0,0) = 0$ is neither a local maximum nor a local minimum (saddle point count as correct). At $(1,-1)$ and $(-1,1)$, $f_{xx} = 12 > 0$, therefore both $f(1,-1) = -2$ and $f(-1,1) = -2$ are local minima. (**2 pts**)

11. (10 pts) Find the Taylor series generated by $\sin^{-1} x$, centered at 0.

Ans:

$$\sin^{-1} x = \int_0^x (1-t^2)^{\frac{-1}{2}} dt \ (\textbf{2pts})$$

$$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-t^2)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(-t^2)^3 + \cdots + +(-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(-t^2)^n + \cdots\right) dt \ \textbf{(5pt}$$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots + \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{x^{2n+1}}{2n+1} + \cdots \ \textbf{(3pts)}$$

12. (10 pts) Evaluate $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$ on $|x| < 1$ using computational rules of power series.

Ans:

The value is $\frac{1}{2}$ for $x = 0$. **(2 pts)**.

For $x \neq 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n+2} &= x^{-2}\left(\sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2}\right) \\
&= x^{-2}\left(\sum_{n=0}^{\infty} \int_0^x t^{n+1}\,dt\right) \ \textbf{(3 pts)} \\
&= x^{-2}\left(\int_0^x \sum_{n=0}^{\infty} t^{n+1}\,dt\right) \\
&= x^{-2}\left(\int_0^x \frac{t}{1-t}\,dt\right) \ \textbf{(3 pts)} \\
&= x^{-2}\left(-x - \ln(1-x)\right) = -\frac{1}{x} - \frac{\ln(1-x)}{x^2}. \ \textbf{(2 pts)}
\end{aligned}$$